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On invertible substitutions with two fixed points *

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Abstract Let φ be a primitive substitution on a two-letter alphabet $\{a,b\}$ having two fixed points ξ_a and ξ_b. We show that the substitution φ is invertible if and only if one has $\xi_a = ab\xi$ and $\xi_b = ba\xi$. ***To cite this article: Z.-X. Wen et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 727–731.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur les substitutions inversibles ayant deux points fixes

Résumé On considère une substitution primitive φ sur l'alphabet $\{a,b\}$ ayant deux points fixes ξ_a et ξ_b (commençant respectivement par a et b). Nous montrons que la substitution φ est inversible si et seulement si l'on a $\xi_a = ab\xi$ et $\xi_b = ba\xi$. ***Pour citer cet article : Z.-X. Wen et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 727–731.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soit $S = \{a,b\}$ un alphabet de deux lettres. Nous désignons par S^* et F_2 le monoïde libre et le groupe libre engendrés par S. Posons $S^+ = S^* \setminus \{\varepsilon\}$ où ε est le mot vide et notons S^ω l'ensemble des mots infinis sur S.

Si w est un mot, nous désignons par $|w|$ sa longueur et par $|w|_a$ (resp. $|w|_b$) le nombre de fois que la lettre a (resp. b) figure dans w.

Un mot $v \in S^*$ est dit *facteur* du mot w (ce que l'on note $v \prec w$) s'il existe deux mots u et u' tels qu'on ait $w = uvu'$. Dans le cas où $u = \varepsilon$ (resp. $u' = \varepsilon$), nous disons que v est un *préfixe* (resp. *suffixe*) de w. Les notions de facteur et de préfixe gardent un sens même si w est un mot infini.

Une *substitution* φ sur S est un endomorphisme de S^*. Dans la suite, nous identifierons la substitution φ au couple de mots $(\varphi(a), \varphi(b))$ et nous supposerons toujours $\varphi(a)$ et $\varphi(b)$ différents de ε. Une substitution sur S agit également sur S^ω (par concatenation des images des lettres composant un mot infini).

S'il existe $c \in S$ tel que c soit un préfixe de $\varphi(c)$ et tel que $|\varphi(c)| \geqslant 2$, alors la suite des mots finis $(\varphi^n(c))_{n\geqslant 1}$ converge vers un point fixe $\varphi^\omega(c) \in S^\omega$ de φ.

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À chaque substitution φ sur S, on associe la matrice M_φ indexée par $S \times S$ dont le coefficient correspondant à (x, y) est le nombre de fois que la lettre x figure dans $\varphi(y)$. Si la matrice M_φ est primitive, la substitution φ est également dite *primitive*.

Une substitution φ est dite inversible si elle définit un automorphisme de F_2.

Notre résultat principal donne la structure des points fixes d'une substitution primitive inversible :

THÉORÈME 1. – *Si ξ_a et ξ_b sont deux points fixes d'une substitution primitive inversible, alors nous avons $\xi_a = ab\xi$ et $\xi_b = ba\xi$. De plus, les trois suites $b\xi$, $a\xi$ et ξ sont points fixes de substitutions primitives inversibles. Réciproquement si les points fixes ξ_a et ξ_b d'une substitution φ sont de la forme $\xi_a = ab\xi$ et $\xi_b = ba\xi$, alors φ est une substitution inversible.*

1. Introduction

Let $S = \{a, b\}$ be a two-letter alphabet. Let S^* (resp. F_2) be the free monoid (resp. the free group) with empty word ε as neutral element generated by S. For any $u, v \in S^*$, we call u a prefix of v (and then we write $u \prec v$) if $v = uw$ for some $w \in S^*$.

If $w \in S^*$ is a word, $|w|$ denotes its *length* and $|w|_a$ (resp. $|w|_b$) the number of times the letter a (resp. b) appears in the word w.

An endomorphism φ of S^* is called a substitution over S. We identify a substitution φ with the couple $(\varphi(a), \varphi(b))$.

One associates with a substitution φ over S a $S \times S$-matrix M_φ: its entry corresponding to $x,\ y$ is the number of times that x appears in $\varphi(y)$. When M_φ is primitive (i.e., one of its power has only positive coefficients), the substitution φ is also said to be *primitive*.

A substitution φ is said to be invertible if it defines an automorphism of F_2. It is proved in [7] that the set $\mathcal{G}$ of invertible substitutions is generated as a monoid by the three following substitutions $\sigma = (ab,\ a)$, $\tau = (ba, a)$ and $\pi = (b, a)$. We define two subclasses of $\mathcal{G}$: $\mathcal{G}_\sigma = \langle \sigma, \pi \rangle$, generated by σ and π, and $\mathcal{G}_\tau = \langle \tau, \pi \rangle$.

It is known [5] that one has $\varphi(bab^{-1}a^{-1}) = u(bab^{-1}a^{-1})^{\pm 1}u^{-1}$ (where $u \in S^*$ or $u^{-1} \in S^*$) for any invertible substitution φ. Furthermore [2], φ is an invertible substitution if and only if $\varphi(bab^{-1}a^{-1}) = u(bab^{-1}a^{-1})^{\pm 1}u^{-1}$ for some $u \in S^*$ or $u^{-1} \in S^*$.

Invertible substitutions and their properties has been studied by many authors (for instance, see Berstel [1], Mignosi [3], Mignosi and Séébold [4], Séébold [6], Wen and Wen [7–9]). They studied both combinatorial and arithmetic properties of fixed points of primitive invertible substitutions. In this note, we give a thorough study of the structure of fixed points of primitive invertible substitutions.

2. Results and proofs

Let $\varphi \in \mathcal{G}_\sigma$, then $\varphi(a)$ and $\varphi(b)$ begin by the same letter, so φ has only one fixed point.

Now consider $\varphi \in \mathcal{G}_\tau$ and we write $\varphi = \pi^\alpha \circ \tau^{n_1} \circ \pi \circ \tau^{n_2} \circ \cdots \circ \pi \circ \tau^{n_k} \circ \pi^\beta$. We call the number $L(\varphi) = \Sigma_{i=1}^{k} n_i + (k-1) + \alpha + \beta$ the *length* of φ.

It is easy to see that, if $L(\varphi)$ is even, then a and b are prefixes of $\varphi(a)$ and $\varphi(b)$ respectively. Therefore, in this case, the substitution φ has two fixed points ξ_a and ξ_b (beginning by a and b respectively).

When $L(\varphi)$ is odd, then a and b are prefixes of $\varphi(b)$ and $\varphi(a)$ respectively, so φ has no fixed point.

Finally we consider $\varphi \notin \mathcal{G}_\tau$. Remark that for any $\varphi \in \mathcal{G}$, we can write $\varphi = \varphi_1 \cdots \varphi_n$ with $\varphi_i \in \{\pi, \sigma, \tau\}$. Since $\varphi \notin \mathcal{G}_\tau$, there exists at least $\varphi_i = \sigma$. Then the first letters of $\varphi(a)$ and $\varphi(b)$ coincide, hence in this case, φ has only one fixed point.

Denote by $\mathcal{G}_{\tau,e}$ and $\mathcal{G}_{\tau,o}$ the subclass of elements of $\mathcal{G}_\tau$ of length even and odd respectively. The following lemma sums up the preceding discussion.

LEMMA 1. – *Let φ be a primitive invertible substitution. Then*
(1) *φ has two fixed points if and only if $\varphi \in \mathcal{G}_{\tau,e}$,*
(2) *φ has only one fixed point if and only if $\varphi \notin \mathcal{G}_{\tau}$,*
(3) *φ has no fixed point if and only if $\varphi \in \mathcal{G}_{\tau,o}$.*

COROLLARY 1. – *Let ξ_a, ξ_b be fixed points of $\varphi \in \mathcal{G}_{\tau,e}$. Then $\xi_a = ab\eta_1$ and $\xi_b = ba\eta_2$.*

Proof. – We have $ab \prec \tau^2(ab)$, $ab \prec \tau \circ \pi(ab)$, $ab \prec \pi \circ \tau(ab)$, $ba \prec \tau^2(ba)$, $ba \prec \tau \circ \pi(ba)$, and $ba \prec \pi \circ \tau(ba)$. This allows a proof by induction on the length of φ.

Now we characterize the elements in $\mathcal{G}_{\tau}$

PROPOSITION 1. – *Let φ be a substitution. Then*
(1) *if $\varphi(aba^{-1}b^{-1}) = aba^{-1}b^{-1}$, then $\varphi \in \mathcal{G}_{\tau,e}$,*
(2) *if $\varphi(aba^{-1}b^{-1}) = (aba^{-1}b^{-1})^{-1}$, then $\varphi \in \mathcal{G}_{\tau,o}$,*
(3) *if $\varphi \in \mathcal{G}_{\tau}$, then $\varphi(aba^{-1}b^{-1}) = (aba^{-1}b^{-1})^{\pm 1}$.*

Proof. – Assertions (1) and (2) follow from $\pi(aba^{-1}b^{-1}) = \tau(aba^{-1}b^{-1}) = (aba^{-1}b^{-1})^{-1}$.

Assume $\varphi(aba^{-1}b^{-1}) = (aba^{-1}b^{-1})^{\pm 1}$. By Ei–Ito's theorem [2], we have $\varphi \in \mathcal{G}$. But $a \prec \varphi(a)$ and $b \prec \varphi(b)$ (or $a \prec \varphi(b)$ and $b \prec \varphi(a)$). Thus by Lemma 1, $\varphi \in \mathcal{G}_{\tau}$.

THEOREM 1. – *Let ξ_a and ξ_b be the fixed points of $\varphi \in \mathcal{G}_{\tau,e}$, then there exists η such that $\xi_a = ab\eta$ and $\xi_b = ba\eta$.*

Proof. – We know (Corollary 1) that $\xi_a = ab\eta$. We have $\varphi(ab\eta) = ab\eta$. Then by Proposition 1, we obtain:

$$\varphi(ba\eta) = \varphi\big(bab^{-1}a^{-1}ab\eta\big) = \varphi\big(bab^{-1}a^{-1}\big)ab\eta = bab^{-1}a^{-1}ab\eta = ba\eta.$$

This means that $ba\eta$ is also a fixed point of φ. It follows that $\xi_b = ba\eta$.

The next theorem shows that the converse holds.

THEOREM 2. – *Suppose ξ_a and ξ_b are two fixed points of the primitive substitution φ with $\xi_a = ab\xi$, $\xi_b = ba\xi$, then $\varphi \in \mathcal{G}_{\tau,e}$.*

Proof. – By hypothesis, we have $\varphi(ab\xi) = ab\xi$ and $\varphi(ba\xi) = ba\xi$. However $|\varphi(ab)| = |\varphi(ba)|$. So there exists $w \in S^*$ such that $\varphi(ab) = abw$ and $\varphi(ba) = baw$. Hence $\varphi(aba^{-1}b^{-1}) = \varphi(ab)\varphi(a^{-1}b^{-1}) = aba^{-1}b^{-1}$, thus by Proposition 1, $\varphi \in \mathcal{G}_{\tau,e}$.

Let $\varphi \in \mathcal{G}_{\tau,e}$ be a primitive invertible substitution, by Theorem 1, φ has two fixed points $ab\eta$ and $ba\eta$. This raises a natural question: are the infinite words $a\eta$, $b\eta$ and η fixed points of primitive invertible substitutions? The answer is given by Theorem 3 below.

LEMMA 2. – *Let $\varphi \in \mathcal{G}_{\tau,e}$ be a primitive substitution and let φ_0, φ_a and φ_b be defined as*:

$$\varphi_0 = \big(b^{-1}a^{-1}\varphi(a)ba,\ a^{-1}b^{-1}\varphi(b)ab\big),$$
$$\varphi_a = \big(b^{-1}\varphi(b)\varphi(a)\varphi(b^{-1})b,\ b^{-1}\varphi(b)b\big),$$
$$\varphi_b = \big(a^{-1}\varphi(a)a,\ a^{-1}\varphi(a)\varphi(b)\varphi(a^{-1})a\big),$$

then φ_b, φ_a, φ_0 are invertible primitive substitutions.

Proof. – We only consider φ_0, since the proof of the assertions for φ_a and φ_b follows the same lines.

Since φ is primitive, $\varphi \neq \pi \circ \tau$. One can easily see that ab and ba are prefix of $\varphi(a)$ and $\varphi(b)$, hence φ_0 is a substitution over S. The primitivity of φ_0 comes from the primitivity of φ.

An easy calculation, using the relation $\varphi(aba^{-1}b^{-1}) = aba^{-1}b^{-1}$, gives

$$\varphi_0\big(aba^{-1}b^{-1}\big) = \big(a^{-1}b^{-1}\varphi(b)\varphi(a)\big)\big(aba^{-1}b^{-1}\big)\big(a^{-1}b^{-1}\varphi(b)\varphi(a)\big)^{-1},$$

which yields $\varphi_0 \in \mathcal{G}$ due to Ei–Ito's theorem.

THEOREM 3. – *Let φ be a substitution in $\mathcal{G}_{\tau,e}$, and $ab\eta$ and $ba\eta$ its two fixed points. Then we have $\varphi_b(b\eta) = b\eta$, $\varphi_a(a\eta) = a\eta$, and $\varphi_0(\eta) = \eta$, where φ_a, φ_b, φ_0 are defined as in Lemma* 2.

Proof. – We only prove $\varphi_0(\eta) = \eta$. The other two cases can be proved in a similar way.

First by using the definition of φ_0 and taking into account the relation $\varphi(aba^{-1}b^{-1}) = aba^{-1}b^{-1}$, we obtain:

$$\varphi_0(a) = \big(a^{-1}b^{-1}\varphi(b)\varphi(a)\big)\varphi(a)\big(a^{-1}b^{-1}\varphi(b)\varphi(a)\big)^{-1},$$
$$\varphi_0(b) = \big(a^{-1}b^{-1}\varphi(b)\varphi(a)\big)\varphi(b)\big(a^{-1}b^{-1}\varphi(b)\varphi(a)\big)^{-1},$$

from which it follows that, for any $w \in S^*$, we have:

$$\varphi_0(w) = \big(a^{-1}b^{-1}\varphi(b)\varphi(a)\big)\varphi(w)\big(a^{-1}b^{-1}\varphi(b)\varphi(a)\big)^{-1}. \tag{$*$}$$

By taking for w longer and longer prefixes of η, we get:

$$\varphi_0(\eta) = a^{-1}b^{-1}\varphi(b)\varphi(a)\varphi(\eta) = a^{-1}b^{-1}\varphi(ba\eta) = \eta.$$

Now, we are going to show some further relations between φ_0 and φ.

Let T be the endomorphism of $\mathcal{G}$ such that $T(\sigma) = \tau$, $T(\tau) = \sigma$, and $T(\pi) = \pi$. This means that, if $\eta = \eta_1 \circ \eta_2 \circ \cdots \circ \eta_n$ with $\eta_i \in \{\sigma, \tau, \pi\}$ $(1 \leqslant i \leqslant n)$, then

$$T\eta = T\eta_1 \circ T\eta_2 \circ \cdots \circ T\eta_n. \tag{1}$$

THEOREM 4. – *For any $\varphi \in \mathcal{G}_{\tau,e}$, we have $\varphi_0 = T\varphi$.*

In other words, φ_0 is obtained by exchanging τ and σ in φ.

Proof. – We have to prove

$$T\varphi = \big(b^{-1}a^{-1}\varphi(a)ba,\ a^{-1}b^{-1}\varphi(b)ab\big). \tag{2}$$

This is done by induction on the length of φ.

First suppose that the length of φ is 2. Since neither $\tau\pi$ nor $\pi\tau$ are primitive, $\varphi = \tau^2$. Hence $\sigma^2 = (b^{-1}a^{-1}\tau^2(a)ba,\ a^{-1}b^{-1}\tau^2(b)ab)$, i.e., the conclusion holds for $k = 1$.

Now assume the conclusion holds when $L(\varphi) = 2k$ and consider the case $L(\varphi) = 2(k+1)$.

Write $\varphi = \psi_2 \circ \psi_1$, where $\psi_1 \in \{\tau^2, \tau \circ \pi, \pi \circ \tau\}$, so $L(\psi_1) = 2$, $L(\psi_2) = 2k$. Now we distinguish three cases depending on ψ_1.

(i) $\psi_1 = \tau^2 = (aba, ba)$. Since $\psi_2 \in \mathcal{G}_{\tau,e}$, we have $\psi_2(aba^{-1}b^{-1}) = aba^{-1}b^{-1}$. From this equality and the hypothesis of induction, we get:

$$\begin{aligned}
T\varphi(a) &= T(\psi_2) \circ T\big(\tau^2\big)(a) = T(\psi_2) \circ \sigma^2(a) = T\psi_2(aba)\\
&= \big(b^{-1}a^{-1}\psi_2(a)ba\big)\big(a^{-1}b^{-1}\psi_2(b)ab\big)\big(b^{-1}a^{-1}\psi_2(a)ba\big)\\
&= b^{-1}a^{-1}\psi_2(aba)ba\\
&= b^{-1}a^{-1}\psi_2 \circ \psi_1(a)ba = b^{-1}a^{-1}\varphi(a)ba,\\
T\varphi(b) &= T\psi_2 \circ \sigma^2(b) = T\psi_2(ab)\\
&= \big(b^{-1}a^{-1}\psi_2(a)ba\big)\big(a^{-1}b^{-1}\psi_2(b)ab\big) = b^{-1}a^{-1}\psi_2(a)\psi_2(b)ab
\end{aligned}$$

$$\begin{aligned}&= a^{-1}b^{-1}\psi_2(b)\psi_2(a)ab = a^{-1}b^{-1}\psi_2(ba)ab\\&= a^{-1}b^{-1}\psi_2\circ\psi_1(b)ab = a^{-1}b^{-1}\varphi(b)ab.\end{aligned}$$

(ii) $\psi_1 = \pi\circ\tau = (ab, b)$. Proceeding as in (i), we have:

$$\begin{aligned}T\varphi(a) &= T\psi_2\circ\pi\circ\sigma(a) = T\psi_2(ba)\\&= \left(a^{-1}b^{-1}\psi_2(b)ab\right)\left(b^{-1}a^{-1}\psi_2(a)ba\right) = a^{-1}b^{-1}\psi_2(b)\psi_2(a)ba\\&= a^{-1}b^{-1}\psi_2\circ\psi_1(a)ba = b^{-1}a^{-1}\varphi(a)ba,\\T\varphi(b) &= T\psi_2\circ\pi\circ\sigma(b) = T\psi_2(b)\\&= a^{-1}b^{-1}\psi_2(b)ab = a^{-1}b^{-1}\varphi(b)ab.\end{aligned}$$

(iii) $\psi_1 = \tau\circ\pi = (a, ba)$. As in (i) we have:

$$\begin{aligned}T\varphi(a) &= T\psi_2\circ\sigma\circ\pi(a) = T\psi_2(a)\\&= b^{-1}a^{-1}\psi_2(a)ba = b^{-1}a^{-1}\varphi(a)ba,\\T\varphi(b) &= T\psi_2\circ\sigma\circ\pi(b) = T\psi_2(ab)\\&= b^{-1}a^{-1}\psi_2(a)baa^{-1}b^{-1}\psi_2(b)ab = b^{-1}a^{-1}\psi_2(a)\psi_2(b)ab\\&= a^{-1}b^{-1}\psi_2(b)\psi_2(a)ab = a^{-1}b^{-1}\varphi(b)ab.\end{aligned}$$

This completes the proof of Theorem 4.

Remark 1. – Let $\varphi = \tau^2$, then τ has two fixed points $\xi_a = ab\xi$ and $\xi_b = ba\xi$. Due to Theorems 3 and 4, we know that ξ is the Fibonacci sequence. In [10] we show that for any $u \in S^*$, $u \neq \varepsilon$,

(1) neither $uab\xi$ nor $uba\xi$ can be a fixed points of an invertible substitution,

(2) $u^{-1}\xi$ cannot be a fixed point of an invertible substitution.

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