C. R. Acad. Sci. Paris, Ser. I 334 (2002) 71–76

Analyse numérique/*Numerical Analysis*

Integral method for the Stokes problem

Toufic Abboud [a], Michel Salaun [b], Stéphanie Salmon [b,c]

[a] **CMAP – École polytechnique, 91128 Palaiseau, France**

[b] **Chaire de calcul scientifique, CNAM UPRES EA n° 3196, 15, rue Marat, 78210 Saint-Cyr-L'École, France**

[c] **INRIA, Projet M3N, Rocquencourt BP 105, 78153 Le Chesnais cedex, France**

Received 14 March 2001; revised and accepted 26 November 2001

Note presented by Olivier Pironneau.

Abstract We consider the bidimensional Stokes problem for incompressible fluids in stream function-vorticity formulation. For this problem, the classical finite elements method of degree one converges only in $\mathcal{O}(\sqrt{h})$ for the quadratic norm of the vorticity, if the domain is convex and the solution regular. We propose to use harmonic functions obtained by a simple layer potential to approach vorticity along the boundary. Numerical results are very satisfying and we prove that this new numerical scheme leads to an error of order $\mathcal{O}(h)$ for the natural norm of the vorticity and under more regularity assumptions from $\mathcal{O}(h^{3/2})$ to $\mathcal{O}(h^2)$ for the quadratic norm of the vorticity. To cite this article: T. Abboud et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 71–76. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Méthode intégrale pour le problème de Stokes

Résumé On considère le problème de Stokes bidimensionnel pour les fluides incompressibles en formulation fonction courant-tourbillon. La méthode des éléments finis classiques de degré un pour les deux champs donne une convergence très lente pour le tourbillon avec de nombreuses oscillations sur le bord. Nous proposons d'utiliser une base de fonctions harmoniques, obtenues à l'aide d'un potentiel de simple couche, pour approcher le tourbillon au bord du domaine. Nous obtenons théoriquement et numériquement avec ce schéma une erreur en moyenne quadratique sur le tourbillon d'ordre 2 par rapport au pas du maillage sous certaines hypothèses de régularité. Pour citer cet article : T. Abboud et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 71–76. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

La formulation variationnelle bien posée [4] du problème de Stokes en fonction courant-tourbillon consiste à chercher le tourbillon ω dans $M(\Omega) = \{\varphi \in \mathrm{L}^2(\Omega),\ \Delta\varphi \in \mathrm{H}^{-1}(\Omega)\}$ et la fonction courant ψ dans $\mathrm{H}_0^1(\Omega)$ satisfaisant le problème (1). Avec un schéma utilisant des éléments finis de degré un pour les deux variables (proposé dans [8]), les estimations d'erreur ne sont pas optimales (*voir* [7]). Nous avons proposé dans [6] de discrétiser directement les deux espaces intervenant dans la décomposition de $M(\Omega)$:

E-mail addresses: abboud@cmapx.polytechnique.fr (T. Abboud); salaun@iat.cnam.fr (M. Salaun); salmon@inria.fr (S. Salmon).

S1631-073X(02)02222-7/FLA

$M(\Omega) = \mathrm{H}^1_0(\Omega) \oplus \mathcal{H}(\Omega)$, où $\mathcal{H}(\Omega) = \{\varphi \in M(\Omega),\ \Delta\varphi = 0\}$. Nous avons montré qu'utiliser des fonctions harmoniques, et non plus des fonctions continues polynomiales de degré 1, pour calculer le tourbillon discret sur le bord du domaine (c'est-à-dire la partie appartenant à l'espace $\mathcal{H}(\Omega)$) améliore les résultats de convergence de la formulation fonction courant-tourbillon : le tourbillon en norme M et la fonction courant en norme H^1 convergent comme $h_{\mathcal{T}}$, où $h_{\mathcal{T}}$ est le pas du maillage. Dans [6], nous approchions l'espace $\mathcal{H}(\Omega)$ à l'aide de fonctions harmoniques discrètes définies sur des maillages homothétiques du maillage initial. Ici, nous choisissons une base de fonctions harmoniques données par représentation intégrale. La trace du maillage sur le bord du domaine est un ensemble d'arêtes de triangles auxquelles on associe une fonction caractéristique. L'espace vectoriel engendré par ces fonctions est noté $\mathcal{C}_{\mathcal{T}}$ (*voir* (3)). Puis on définit l'espace $\mathcal{H}_{\mathcal{T},I}$ engendré par le potentiel de simple couche [9] appliqué aux fonctions de $\mathcal{C}_{\mathcal{T}}$ (définition 1). La partie intérieure du tourbillon est toujours approchée par des fonctions continues polynomiales de degré 1 s'annulant au bord. L'espace $M(\Omega)$ discrétisé est donc de même dimension que celui des fonctions continues polynomiales de degré 1 et n'en diffère que par les fonctions choisies au bord. On construit alors en définition 3 un opérateur d'interpolation sur l'espace $\mathcal{H}_{\mathcal{T},I}$. Etant données les erreurs d'interpolation (proposition 2) obtenues, on a le théorème de convergence suivant :

THÉORÈME 1 (Convergence). – *Le problème* (4)–(7) *a une unique solution* $(\psi_{\mathcal{T}}, \omega_{\mathcal{T}}) \in \mathrm{H}^1_{0,\mathcal{T}} \times \mathrm{H}_{\mathcal{T},I}$. *Si le domaine* Ω *est convexe et si* $(\psi,\omega) \in (\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)) \times \mathrm{H}^2(\Omega)$ *est solution du problème continu alors, pour toute triangulation régulière, il existe* $C > 0$, *indépendante du maillage, tel que* :

$$\|\omega - \omega_{\mathcal{T}}\|_{0,\Omega} + \|\psi - \psi_{\mathcal{T}}\|_{0,\Omega} \leqslant C h_{\mathcal{T}}^{3/2} \big(\|\omega\|_{2,\Omega} + \|\psi\|_{2,\Omega}\big).$$

Si de plus $\omega \in \mathrm{H}^{5/2}(\Omega)$, *on a* :

$$\|\omega - \omega_{\mathcal{T}}\|_{0,\Omega} + \|\psi - \psi_{\mathcal{T}}\|_{0,\Omega} \leqslant C h_{\mathcal{T}}^{2} \big(\|\omega\|_{2,\Omega} + \|\psi\|_{2,\Omega}\big).$$

Les résultats numériques sur le test proposé par Bercovier et Engelman [3] sont en accord avec le théorème précédent. La figure 1 montre que la convergence en norme quadratique du tourbillon et de la fonction courant est d'ordre 2 alors que, pour le schéma utilisant des fonctions affines continues pour les deux champs, elle n'est que d'ordre $1/2$ pour le tourbillon en norme L^2 et $1-\varepsilon$ ($\varepsilon > 0$) pour la fonction courant en semi-norme H^1.

1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^2$, with boundary Γ assumed to be polygonal. Let also $\mathbf{f}$ be a field of given forces, belonging to $(\mathrm{L}^2(\Omega))^2$, whose standard inner product is denoted by $(\cdot,\cdot)$. The well-posed weak formulation [4] of the stationary and homogeneous Stokes problem needs the introduction of the classical functional space $\mathrm{H}^1_0(\Omega)$, its dual $\mathrm{H}^{-1}(\Omega)$ ($\langle\cdot,\cdot\rangle_{-1,1}$ shall denote the duality product between $\mathrm{H}^{-1}(\Omega)$ and $\mathrm{H}^1_0(\Omega)$) and the space $M(\Omega)$ defined by: $M(\Omega) = \{\varphi \in \mathrm{L}^2(\Omega),\ \Delta\varphi \in \mathrm{H}^{-1}(\Omega)\}$. Space $M(\Omega)$ is a Hilbert space equipped with the norm $\|\varphi\|_M = \sqrt{\|\varphi\|^2_{0,\Omega} + \|\Delta\varphi\|^2_{-1,\Omega}}$ [4]. We introduce the left kernel $\mathcal{H}(\Omega)$ of the bilinear form $\langle\Delta\cdot,\cdot\rangle_{-1,1}$: $\mathcal{H}(\Omega) = \{\varphi \in M(\Omega),\ \Delta\varphi = 0\}$. Moreover, we recall the decomposition: $M(\Omega) = \mathrm{H}^1_0(\Omega) \oplus \mathcal{H}(\Omega)$. This decomposition is established by splitting $\varphi \in M(\Omega)$ into two parts: $M(\Omega) \ni \varphi \mapsto (\varphi^0, \varphi^\Delta) \in \mathrm{H}^1_0(\Omega) \times \mathcal{H}(\Omega)$. The component φ^0 is the unique variational solution in $\mathrm{H}^1_0(\Omega)$ of the Dirichlet problem $\Delta\varphi^0 = \Delta\varphi$ in Ω, $\gamma\varphi^0 = 0$ on Γ (where γ denotes the trace operator from $\mathrm{H}^1(\Omega)$) and $\varphi^\Delta = \varphi - \varphi^0$ is a harmonic function. The variational formulation of the Stokes problem in stream function-vorticity consists in finding ψ in $\mathrm{H}^1_0(\Omega)$ and ω in $M(\Omega)$ such that:

$$\begin{cases} (\omega,\varphi) + \langle\Delta\varphi,\psi\rangle_{-1,1} = 0, & \forall\varphi \in M(\Omega), \\ \langle -\Delta\omega,\xi\rangle_{-1,1} = (\mathbf{f}, \mathbf{rot}\,\xi), & \forall\xi \in \mathrm{H}^1_0(\Omega). \end{cases} \tag{1}$$

Let us recall that problem (1) is well-posed [4]. Using the above-mentioned decomposition of $M(\Omega)$, it is obvious that the previous problem is equivalent to:

$$\begin{cases} \text{find } \omega^0 \in \mathrm{H}_0^1(\Omega)\text{:} \quad \langle -\Delta\omega^0, \xi\rangle_{-1,1} = (\nabla\omega^0, \nabla\xi) \;=\; (\mathbf{f}, \mathbf{rot}\,\xi), & \forall \xi \in \mathrm{H}_0^1(\Omega), \\ \text{find } \omega^\Delta \in \mathcal{H}(\Omega)\text{:} \qquad\qquad\qquad\qquad (\omega^\Delta, \varphi) \;=\; -(\omega^0, \varphi), & \forall \varphi \in \mathcal{H}(\Omega), \\ \text{find } \psi \in \mathrm{H}_0^1(\Omega)\text{:} \quad \langle -\Delta\chi, \psi\rangle_{-1,1} = (\nabla\psi, \nabla\chi) \;=\; (\omega^0 + \omega^\Delta, \chi), & \forall \chi \in \mathrm{H}_0^1(\Omega). \end{cases} \tag{2}$$

Error estimates derived for the linear finite element scheme used to approximate (2) are not optimal. If $h_\mathcal{T}$ denotes the maximum diameter of the elements of a triangulation, the bound is in $h_\mathcal{T}^{1/2}$ for the L^2-norm of the vorticity if the domain is convex and the solution regular [7]. A recent work [1] shows that it is possible to stabilize the numerical scheme by adding jumps at interfaces and prove convergence. Our approach consists in a direct discretization of the decomposition of $M(\Omega)$. A previous work of authors used with success homothetic mesh refinements to solve the second step of (2) (Dubois, Salaün, Salmon [6]). The following section gives an alternative way to solve this problem by a direct use of harmonic functions, based on integral representation. This last idea was previously developped by Achdou and Pironneau [2] but in our case, only the harmonic part of the vorticity is obtained by integral method.

2. Discretization using harmonic functions

2.1. Discretization of space $\mathcal{H}$

We recall that Ω being polygonal allows to cover it entirely with a mesh $\mathcal{T}$ assumed to belong to a regular family of triangulations in the sense of Ciarlet [5]. We introduce the trace of mesh $\mathcal{T}$ on the boundary Γ. It is a set $\mathcal{A}(\mathcal{T},\Gamma)$ of edges of triangles of the mesh which are contained in Γ. If $N_a(\mathcal{T},\Gamma)$ is the number of these edges, we denote them by $\Gamma_i,\ 1 \leqslant i \leqslant N_a(\mathcal{T},\Gamma)$. As Γ is closed, $N_a(\mathcal{T},\Gamma)$ is also equal to the number of vertices of the mesh $\mathcal{T}$ on the boundary Γ. Then we define the vectorial space $\mathcal{C}_\mathcal{T}$ generated by characteristic functions of edges $\Gamma_i \in \mathcal{A}(\mathcal{T},\Gamma)$ of Γ:

$$\mathcal{C}_\mathcal{T} = \mathrm{Span}\big\{q_i = \mathbb{1}_{\Gamma_i},\ \Gamma_i \in \mathcal{A}(\mathcal{T},\Gamma)\big\}, \tag{3}$$

where $\mathbb{1}_{\Gamma_i}$ is the characteristic function defined from Γ to $\mathbb{R}$ by: $\mathbb{1}_{\Gamma_i}(x) = \begin{cases} 1 & \text{if } x \in \Gamma_i, \\ 0 & \text{if } x \notin \Gamma_i. \end{cases}$

Dimension of $\mathcal{C}_\mathcal{T}$ is obviously equal to $N_a(\mathcal{T},\Gamma)$. Then we denote by $\mathcal{S}$ the simple layer operator (*see* [9]).

DEFINITION 1. – We define $\mathcal{H}_{\mathcal{T},I}$ the space generated by the functions $\mathcal{S}q_i$, for all $q_i \in \mathcal{C}_\mathcal{T}$: $\mathcal{H}_{\mathcal{T},I} = \mathrm{Span}\{\mathcal{S}q_i,\ q_i \in \mathcal{C}_\mathcal{T}\}$, where $\mathcal{S}q_i(x) = \int_\Gamma G(x,y)q_i(y)\,\mathrm{d}\gamma_y,\ \forall x \in \overline{\Omega}$, and $G(x,y) = \frac{1}{2\pi}\log|x-y|$ is the Green kernel.

Discrete space $\mathcal{H}_{\mathcal{T},I}$ is finite dimensional and, clearly, $\dim \mathcal{H}_{\mathcal{T},I} = \dim \mathcal{C}_\mathcal{T}$. By construction, functions of $\mathcal{H}_{\mathcal{T},I}$ are harmonic.

DEFINITION 2. – For the discretization of $M(\Omega)$, we set: $\mathrm{H}_\mathcal{T}^{1,I} = \mathrm{H}_{0,\mathcal{T}}^1 \oplus \mathcal{H}_{\mathcal{T},I}$, where $\mathrm{H}_{0,\mathcal{T}}^1$ is the set of piecewise linear continuous functions that vanish on the boundary.

Then space $\mathrm{H}_\mathcal{T}^{1,I}$ is almost the same as the space of linear piecewise continuous functions but near the boundary, we use harmonic functions.

2.2. Discrete formulation

We thus propose the following discrete formulation of the Stokes problem based on (2):

$$\psi_{\mathcal{T}} \in \mathrm{H}^1_{0,\mathcal{T}}, \quad \omega_{\mathcal{T}} = \omega^0_{\mathcal{T}} + \omega^\Delta_{\mathcal{T}} \in \mathrm{H}^{1,I}_{\mathcal{T}} = \mathrm{H}^1_{0,\mathcal{T}} \oplus \mathcal{H}_{\mathcal{T},I}, \tag{4}$$

$$\left(\nabla\omega^0_{\mathcal{T}}, \nabla\xi\right) = (\mathbf{f}, \mathbf{rot}\,\xi), \quad \forall\xi \in \mathrm{H}^1_{0,\mathcal{T}}, \tag{5}$$

$$\left(\omega^\Delta_{\mathcal{T}}, \varphi\right) = -\left(\omega^0_{\mathcal{T}}, \varphi\right), \quad \forall\varphi \in \mathcal{H}_{\mathcal{T},I}, \tag{6}$$

$$(\nabla\psi_{\mathcal{T}}, \nabla\chi) = \left(\omega^0_{\mathcal{T}} + \omega^\Delta_{\mathcal{T}}, \chi\right), \quad \forall\chi \in \mathrm{H}^1_{0,\mathcal{T}}. \tag{7}$$

The above method is a conforming discretization of problem (2) since $\mathcal{H}_{\mathcal{T},I}$ is contained in $\mathcal{H}(\Omega)$. The way of studying problem (4)–(7) follows ideas of [8].

PROPOSITION 1. – *Existence and uniqueness of a solution to problem* (4)–(7). *If* $\mathbf{f} \in (\mathrm{L}^2(\Omega))^2$, *problem* (4)–(7) *has a unique solution* $(\psi_{\mathcal{T}}, \omega_{\mathcal{T}}) \in \mathrm{H}^1_{0,\mathcal{T}} \times \mathrm{H}^{1,I}_{\mathcal{T}}$ *which depends continuously on the datum* $\mathbf{f}$. *There exists a strictly positive constant* C *independent of the mesh such that*:

$$\|\omega_{\mathcal{T}}\|_M + \|\nabla\psi_{\mathcal{T}}\|_{0,\Omega} \leqslant C\|\mathbf{f}\|_{0,\Omega}. \tag{8}$$

2.3. Interpolation error

Let us introduce the L^2-projection operator $p_{\mathcal{C}}$ on the space of piecewise constants. For all $\rho \in \mathrm{L}^2(\Gamma)$, $p_{\mathcal{C}}\rho$ is the element of $\mathcal{C}_{\mathcal{T}}$ such that: $\int_\Gamma (p_{\mathcal{C}}\rho - \rho)\cdot q\,\mathrm{d}\gamma = 0$, $\forall q \in \mathcal{C}_{\mathcal{T}}$. Then, let $\Pi_{\mathcal{T}} : \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_{0,\mathcal{T}}$ be the classical Lagrange interpolation operator associated with mesh $\mathcal{T}$ and γ will denote the trace operator.

DEFINITION 3 (Interpolation operator). – The interpolation operator $\phi_{\mathcal{T}} : \mathcal{H}(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathcal{H}_{\mathcal{T},I}$ is defined by $\phi_{\mathcal{T}}\varphi^\Delta = \zeta$ where

$$\zeta(x) = \mathcal{S}p_{\mathcal{C}}\left(\gamma\varphi^\Delta\right) = \int_\Gamma G(x,y)\cdot p_{\mathcal{C}}\left(\gamma\varphi^\Delta\right)(y)\,\mathrm{d}\gamma_y, \quad \forall x \in \overline{\Omega}.$$

We define the interpolation operator $\mathcal{P}_{\mathcal{T}}$ from $M(\Omega) \cap \mathrm{H}^2(\Omega)$ to $\mathrm{H}^{1,I}_{\mathcal{T}} = \mathrm{H}^1_{0,\mathcal{T}} \oplus \mathcal{H}_{\mathcal{T},I}$ by

$$\mathcal{P}_{\mathcal{T}} : M(\Omega) \cap \mathrm{H}^2(\Omega) \ni \varphi = \varphi^0 + \varphi^\Delta \mapsto \mathcal{P}_{\mathcal{T}}\varphi = \Pi_{\mathcal{T}}\varphi^0 + \phi_{\mathcal{T}}\varphi^\Delta \in \mathrm{H}^{1,I}_{\mathcal{T}}.$$

PROPOSITION 2 (Error estimates). – *For* $\mathcal{T}$ *in a regular family of triangulations and for* φ *in* $M(\Omega)$ *splitted into* φ^0 *and* φ^Δ, *if we suppose* $\varphi^0 \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ *and* $\varphi \in \mathrm{H}^2(\Omega)$, *then there exists some strictly positive constants, say* C, *only dependent on the mesh family, such that*:

$$\left\|\varphi^0 - \Pi_{\mathcal{T}}\varphi^0\right\|_M \leqslant C h_{\mathcal{T}}|\varphi^0|_{2,\Omega}, \tag{9}$$

$$\left\|\varphi^\Delta - \phi_{\mathcal{T}}\varphi^\Delta\right\|_M \leqslant C h^{3/2}_{\mathcal{T}}\|\varphi\|_{2,\Omega}. \tag{10}$$

Moreover, if φ *belongs to* $\mathrm{H}^{5/2}(\Omega)$, *we have*:

$$\left\|\varphi^\Delta - \phi_{\mathcal{T}}\varphi^\Delta\right\|_M \leqslant C h^2_{\mathcal{T}}\|\varphi\|_{5/2,\Omega}. \tag{11}$$

The proof is based on the following result: for all $q \in_\circ \mathrm{H}^{-3/2}(\Gamma) = \{\mu \in \mathrm{H}^{-3/2}(\Gamma),\ \langle\mu, 1\rangle_{-3/2,3/2} = 0\}$, we can define the harmonic function $\mathcal{S}q$:

$$\mathcal{S}q(x) = \int_\Gamma G(x,y)q(y)\,\mathrm{d}\gamma_y, \quad \forall x \in \overline{\Omega},$$

such that there exists a constant $C > 0$: $\|\mathcal{S}q\|_{0,\Omega} \leqslant C\|q\|_{-3/2,\Gamma}$.

2.4. Convergence results

With the stability of the scheme and the interpolation errors obtained in Proposition 2, we have:

THEOREM 1 (First convergence result). – *If $\mathcal{T}$ belongs to a regular family of triangulations, discrete problem* (4)–(7) *has a unique solution* $(\psi_{\mathcal{T}}, \omega_{\mathcal{T}}) \in \mathrm{H}^1_{0,\mathcal{T}} \times \mathrm{H}^{1,I}_{\mathcal{T}}$*, associated with problem* (2)*. If* $\omega = \omega^0 + \omega^\Delta$ *is such that* $\omega \in \mathrm{H}^2(\Omega)$ *and* $\omega^0 \in \mathrm{H}^2(\Omega)$*, then there exists* $C > 0$ *such that for all mesh* $\mathcal{T}$:

$$\|\omega - \omega_{\mathcal{T}}\|_M + \|\psi - \psi_{\mathcal{T}}\|_{1,\Omega} \leqslant C h_{\mathcal{T}}\left(\left|\omega^0\right|_{2,\Omega} + \|\omega\|_{2,\Omega} + |\psi|_{2,\Omega}\right).$$

Remark 1. – Let φ be in $M(\Omega)$ splitted into: $\varphi = \varphi^0 + \varphi^\Delta$ with $\varphi^0 \in \mathrm{H}^1_0(\Omega)$ and $\varphi^\Delta \in \mathcal{H}(\Omega)$ (i.e. harmonic). Then, if Ω is supposed *convex* and if φ belongs to $\mathrm{H}^2(\Omega)$, φ^0 and φ^Δ also belong to $\mathrm{H}^2(\Omega)$. Moreover, there exists a constant $C > 0$ such that:

$$\left\|\varphi^0\right\|_{2,\Omega} \leqslant C\|\varphi\|_{2,\Omega} \quad \text{and} \quad \left\|\varphi^\Delta\right\|_{2,\Omega} \leqslant C\|\varphi\|_{2,\Omega}.$$

Thus, if Ω is convex, Theorem 1 becomes: there exists $C > 0$ independent of the mesh family such that:

$$\|\omega - \omega_{\mathcal{T}}\|_M + \|\psi - \psi_{\mathcal{T}}\|_{1,\Omega} \leqslant C h_{\mathcal{T}}\left(\|\omega\|_{2,\Omega} + |\psi|_{2,\Omega}\right).$$

Theorem 1 is important because it shows that using a space of harmonic functions along the boundary gives an error of order $\mathcal{O}(h_{\mathcal{T}})$ when $\omega \in \mathrm{H}^2(\Omega)$, which improves previous known results and is equivalent to those proved in [6].

THEOREM 2 (Second convergence result). – *If* Ω *is supposed convex and under same assumptions as Theorem* 1*, there exists* C*, strictly positive such that for all* $\mathcal{T}$:

$$\|\omega - \omega_{\mathcal{T}}\|_{0,\Omega} + \|\psi - \psi_{\mathcal{T}}\|_{0,\Omega} \leqslant C h_{\mathcal{T}}^{3/2}\left(\|\omega\|_{2,\Omega} + \|\psi\|_{2,\Omega}\right).$$

If moreover $\omega \in \mathrm{H}^{5/2}(\Omega)$*, we have*:

$$\|\omega - \omega_{\mathcal{T}}\|_{0,\Omega} + \|\psi - \psi_{\mathcal{T}}\|_{0,\Omega} \leqslant C h_{\mathcal{T}}^{2}\left(\|\omega\|_{2,\Omega} + \|\psi\|_{2,\Omega}\right).$$

The last part of the previous theorem says that if the solution is more regular than usual, the convergence is of order 2. This is illustrated in the numerical examples in the next section where the solutions are very regular.

3. Numerical results

The first numerical experiments have been performed on a unit square where an analytical solution is completely known (test of Bercovier–Engelman [3]). We have worked with unstructured meshes as structured meshes give good results without any stabilization see [7]. The number of harmonic functions used is equal to the number of vertices on the boundary. Moreover the classical method, which uses linear functions for approximating the vorticity and the stream-function, is not stable: error bounds are in $h_{\mathcal{T}}^{1/2}$ for the L^2-norm of the vorticity [7] (as numerically illustrated on Figure 1). Using constants on edges with regular solutions, we have obtained, as expected by Theorem 2, convergence of order 2 for the L^2-norm of the vorticity and for the L^2-norm of the stream function, see Figure 1.

4. Conclusion

We have studied the well-posed Stokes problem in stream function and vorticity formulation. We have shown that using the space of real harmonic functions is sufficient to obtain convergence with better estimations than the previous ones. We have proposed a way of approaching numerically the space of real harmonic functions by the help of boundary elements which is a large gain of time compared to previous results in [6]. We have shown theoretically and numerically that, by this way, we obtain convergence on the quadratic norm of the vorticity with optimal rate in regular cases.

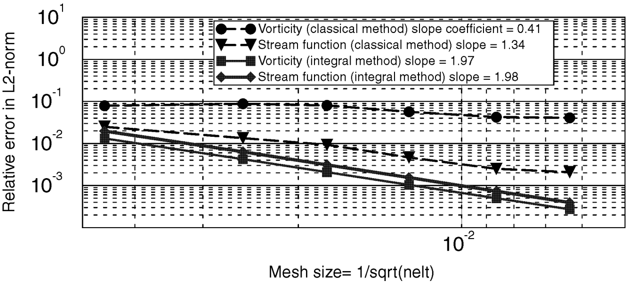

Figure 1. – Convergence orders – Bercovier–Engelman test.

Figure 1. – Ordre de convergence – test de Bercovier–Engelman.

References

[1] Amara M., Bernardi C., Convergence of a finite element discretization of the Navier–Stokes equations in vorticity and stream function formulation, M2AN 33 (5) (1999) 1033–1056.
[2] Achdou Y., Pironneau O., Integral equation for the generalized Stokes operator with applications to boundary layer matching, C. R. Acad. Sci. Paris 315 (1) (1992) 91–96.
[3] Bercovier M., Engelman M., A finite element for the numerical solution of viscous incompressible flows, J. Comput. Phys. 30 (1979) 181–201.
[4] Bernardi C., Girault V., Maday Y., Mixed spectral element approximation of the Navier–Stokes equations in stream function and vorticity formulation, IMA J. Numer. Anal. 12 (1992) 565–608.
[5] Ciarlet P.G., The Finite Element Methods for Elliptic Problems, North-Holland, 1987.
[6] Dubois F., Salaün M., Salmon S., Discrete harmonics for the Stokes problem, C. R. Acad. Sci. Paris 331 (1) (2000) 827–832.
[7] Girault V., Raviart P.-A., Finite Element Method for Navier–Stokes Equations, Springer-Verlag, 1986.
[8] Glowinski R., Pironneau O., Numerical methods for the first biharmonic equation and for the bidimensional Stokes problem, SIAM 21 (1979) 167–211.
[9] Nédélec J.-C., Approximation des équations intégrales en mécanique et en physique, Internal report École Polytechnique, 1977.