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Differential Geometry

Hyperbolic manifolds, amalgamated products and critical exponents

Variétés hyperboliques, produits amalgamés et exposants critiques

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Abstract

We give a new proof of a result due to Y. Shalom: if the fundamental group of a compact real hyperbolic manifold of dim n is a free product of its subgroups A and B over the amalgamated subgroup C, then the critical exponent of C is not smaller than $n-2$. The proof, which is geometric, allows one to treat the equality case and an extension to variable curvature. ***To cite this article: G. Besson et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous donnons une preuve nouvelle d'un résultat dû à Y. Shalom ; précisément, nous montrons que, si le groupe fondamental d'une variété hyperbolique réelle compacte de dimension n est le produit libre de ses sous-groupes A et B amalgamé sur C, alors l'exposant critique de C est plus grand que $n-2$. La preuve, géométrique, permet de traiter le cas d'égalité ainsi qu'une extension au cas de courbure variable. ***Pour citer cet article : G. Besson et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Nous nous intéressons aux variétés hyperboliques réelles compactes dont le groupe fondamental est un produit amalgamé.

Soit X^n une variété hyperbolique réelle compacte de dimension n dont le groupe fondamental Γ vérifie $\Gamma = A *_C B$; de telles variétés existent, par exemple en dimension 3, grâce au théorème d'hyperbolisation de W. Thurston (cf. [4,8]). En toute dimension, A. Lubotsky [5] a montré que tous les exemples arithmétiques

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standards possèdent un revêtement fini dont le groupe fondamental est comme ci-dessus. Y. Shalom a prouvé dans [7] que, dans cette situation, le sous-groupe C doit être « gros ». Précisément, définissons l'exposant critique du sous-groupe discret C du groupe d'isométries hyperboliques comme,

$$\delta(C) = \inf\left\{s > 0 \;\Big|\; \sum_{\gamma \in C} \mathrm{e}^{-sd(\gamma x, x)} < \infty\right\},$$

où $x \in \mathbb{H}^n_{\mathbb{R}}$ est un point quelconque. On vérifie aisément que cette définition ne dépend pas du choix de x.

Théorème 0.1 [7, 1.6]. *Soit Γ un réseau de* $\mathrm{SO}(n,1)$. *Supposons que $\Gamma = A *_C B$ est le produit libre de A et B amalgamé sur C, alors $\delta(C) \geqslant n-2$.*

L'inégalité stricte a lieu si $G \neq \mathrm{SO}(3,1)$ et $[A:C] < \infty$, $[B:C] < \infty$.

Remarque. Y. Shalom donne un résultat similaire pour $\mathrm{SU}(n,1)$ et prouve dans ce cas une inégalité stricte.

Si X^n contient une hypersurface compacte totalement géodésique Y^{n-1} qui sépare X^n en deux composantes connexes, alors $\Gamma =: \pi_1 X^n$ est amalgamé sur $C =: \pi_1 Y^{n-1}$ et $\delta(C) = n-2$. Y. Shalom suggère que l'égalité dans le théorème précédent ne se produit que dans ce cas.

Dans cette Note, nous donnons une preuve géométrique du Théorème 0.1, pour les réseaux uniformes de $\mathrm{SO}(n,1)$ et pour $n \geqslant 4$, qui conduit au cas d'égalité et peut être étendue au cas où la courbure est variable.

Théorème 0.2. *Soit X^n, $n \geqslant 4$, une variété hyperbolique compacte de dimension n dont le groupe fondamental vérifie $\Gamma = A *_C B$. Supposons, pour simplifier, que $[\Gamma : A]$ et $[\Gamma : B]$ sont infinis, alors*

$$\delta(C) \geqslant n-2 = \delta(\Gamma) - 1.$$

L'égalité a lieu si et seulement si X contient une hypersurface compacte totalement géodésique dont C est le groupe fondamental.

Nous tenons à remercier Jean Barge, Gaël Collinet et Jean Lannes pour des conversations fructueuses et Gilles Carron pour nous avoir expliqué une preuve du Théorème 0.2 utilisant la cohomologie L^2.

1. Introduction

In this Note we shall be interested in a compact hyperbolic manifold whose fundamental group is an amalgamated product.

Notation. Let X^n be a compact hyperbolic manifold with fundamental group Γ. We assume $\Gamma = A *_C B$ is a free product of its subgroups A and B over the amalgamated subgroup C. Such manifolds do exist, for example in dimension 3 thanks to the Thurston's hyperbolization theorem, cf. [4,8]. In any dimension, A. Lubotsky, [5] showed that any standard arithmetic example possesses finite coverings whose fundamental groups are as above. In [7], Theorem 1.6, Y. Shalom showed that, in this situation, the subgroup C has to be "big". More precisely, let us define the critical exponent of a discrete subgroup C of isometries of the hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ as

$$\delta(C) = \inf\left\{s > 0 \;\Big|\; \sum_{\gamma \in C} \mathrm{e}^{-sd(\gamma x, x)} < \infty\right\},$$

where $x \in \mathbb{H}^n_{\mathbb{R}}$ is an arbitrary point. One can easily check that this definition does not depend on the chosen point x.

Theorem 1.1 [7, 1.6]. *Let Γ be a lattice in* $\mathrm{SO}(n,1)$. *Assume that $\Gamma = A *_C B$ is a free product of its subgroups A and B amalgamated over C, then $\delta(C) \geqslant n-2$.*

Strict inequality holds if $G \neq \mathrm{SO}(3,1)$ *and* $[A:C] < \infty$, $[B:C] < \infty$.

Remark. Y. Shalom gives a similar result for lattices in $\mathrm{SU}(n,1)$ and also proves strict inequality in this situation.

Clearly, if X^n contains a compact totally geodesic hypersurface Y^{n-1} which separates X^n in two connected components, then $\Gamma =: \pi_1 X^n$ is a free product amalgamated over $C =: \pi_1 Y^{n-1}$ and $\delta(C) = n-2$. Shalom suggests that equality in Theorem 1.1 should occur only in this situation.

In this Note we shall give a geometrical proof of Theorem 1.1 for uniform lattices in $\mathrm{SO}(n,1)$, for $n \geqslant 4$, which yields the equality case and can be extended to the variable curvature case.

Theorem 1.2. *Let* X^n, $n \geqslant 4$, *be a compact* n*-dimensional hyperbolic manifold whose fundamental group* $\Gamma = A *_C B$. *Let us assume, for simplicity, that* $[\Gamma : A]$ *and* $[\Gamma : B]$ *are infinite. Then*

$$\delta(C) \geqslant n-2 = \delta(\Gamma) - 1.$$

The equality occurs if and only if X *contains a compact totally geodesic hypersurface with* C *as fundamental group.*

Remarks.

(i) In the case where $[\Gamma : A]$ or $[\Gamma : B]$ is finite then either $\delta(C) = n-1$ or the decomposition is trivial, i.e., one factor A or B is equal to C in which case C can be arbitrary.
(ii) Following Serre's theorem (see [6, p. 49]), Γ acts on a (simplicial) tree with a segment as fundamental domain, where A and B are the stabilizers of the two vertices respectively and C is the stabilizer of the edge. Theorem 1.6 in [7] is stated for Γ acting on a tree without a fixed vertex; this contains the above situation and also the case of graph of groups. We shall treat such extensions in a forthcoming paper.

We wish to thank Jean Barge, Gaël Collinet and Jean Lannes for fruitful exchanges and Gilles Carron for explaining to us his proof of Theorem 1.2 using L^2-cohomology.

Sketch of proof of Theorem 1.2. *Step* 1: We show that there exists a smooth hypersurface $\widetilde{Z} \subset \mathbb{H}^n$ which is C-invariant and such that, for a subgroup C' of C, $Z = \widetilde{Z}/C'$ is a compact essential submanifold of $\mathbb{H}^n/C'$, namely it has the property that the natural morphism $\mathbb{R} \cong H_{n-1}(Z,\mathbb{R}) \to H_{n-1}(\mathbb{H}^n/C',\mathbb{R})$ is nonzero.

The hypersurface $\widetilde{Z}$ is obtained as a connected component of the inverse image by f of a regular value, where $f : \mathbb{H}^n \to T$ is a Γ-equivariant map. Here T is the tree coming from Serre's theorem. Then we check that Z separates $\mathbb{H}^n/C'$ in two noncompact components which yields the nontriviality of the above morphism.

Step 2:

Lemma 1.3. *There exists* $V > 0$ *such that, for any hypersurface* $\widetilde{Z} \subset \mathbb{H}^n$ *which is* C'*-invariant and such that* $Z = \widetilde{Z}/C'$ *is essential in* $\mathbb{H}^n/C'$, *then* $\mathrm{vol}_{n-1}(Z') \geqslant V > 0$.

Proof. This is a consequence of an isosystolic inequality due to M. Gromov [3, p. 66]. Precisely, for an essential compact submanifold Z in $\mathbb{H}^n/C'$ and for any metric g on Z, we have the following

$$\mathrm{vol}(Z,g) \geqslant C_n \big(\inf\{d_{\tilde{g}}(z,\gamma z) \mid z \in \widetilde{Z}, \gamma \in C'\}\big)^{n-1},$$

where $d_{\tilde{g}}$ is the induced distance on $\widetilde{Z}$ and C_n is a constant depending on the dimension only. □

Step 3: The fact that $H_{n-1}(\mathbb{H}^n/C,\mathbb{R})$ is not zero and that C contains only hyperbolic isometries shows that C is nonelementary. Consequently C possesses a (nonunique) Patterson–Sullivan measure. For the sake of simplicity let us assume that one of them contains no atom and let us call it μ_x for $x \in \mathbb{H}^n$. We then define the natural map (see [1]) by $\widetilde{F}(x) = \text{barycenter}(\mu_x)$.

This map is naturally defined from $\mathbb{H}^n$ to $\mathbb{H}^n$, is C'-equivariant and satisfies (see [1])

$$\forall x \in \mathbb{H}^n, \quad \mathrm{Jac}_{n-1}\big(\widetilde{F}(x)\big) \leqslant \left(\frac{\delta(C)}{n-2}\right)^{n-1},$$

where Jac_{n-1} is the $(n-1)$-dimensional Jacobian. Therefore $\widetilde{F}$ defines a map $F : \mathbb{H}^n/C' \to \mathbb{H}^n/C'$ such that for any compact hypersurface Z,

$$\mathrm{vol}_{n-1}\big(F^k(Z)\big) \leqslant \left(\frac{\delta(C)}{n-2}\right)^{k(n-1)} \mathrm{vol}(Z),$$

where $F^k = F \circ \cdots \circ F$ (k-times).

Now, if $\delta(C) < n-2$ then $\mathrm{vol}(F^k(Z)) \underset{k\to+\infty}{\longrightarrow} 0$ which contradicts step 2. This shows the inequality in Theorem 1.2.

In the case where the Patterson–Sullivan measure has atoms we just approximate it by a family of atomless measure satisfying the same kind of properties (see [1]).

Step 4: In the equality case, $\delta(C) = n-2$. We prove the following

Lemma 1.4. *There exists $x_0 \in \mathbb{H}^n/C'$ such that $F(x_0) = x_0$; moreover, for such an x_0, $D\widetilde{F}(\tilde{x}_0)$ (for all $\tilde{x}_0 \in \mathbb{H}^n$ which projects on x_0) is an orthogonal projector on some hyperplane E in $T_{\tilde{x}_0}\mathbb{H}^n$ and the limit set Λ_C of C is included in $E(\infty)$, the boundary at infinity of the totally geodesic hypersurface H generated by E.*

Sketch of proof. For $x \in \mathbb{H}^n$, such that $\mathrm{Jac}_{n-1}\widetilde{F}(x) = 1$ then $\widetilde{F}(x) = x$, and the limit set Λ_C of C is included in $E(\infty)$ where E is the hyperplane of $T_x\mathbb{H}^n$ such that $\mathrm{Jac}_{n-1}\widetilde{F}(x) = |\det(D\widetilde{F}(x)_{|E})|$. This relies on the properties of the natural map as described in [1]. The lemma now reduces to finding a point $x_0 \in \mathbb{H}^n/C'$ such that $\mathrm{Jac}\, F(x_0) = 1$. For this purpose we need the following lemma which follows from a result due to Carron and Pedon [2].

Lemma 1.5. *There exists a L^2-harmonic $(n-1)$-form on $\mathbb{H}^n/C'$ such that $\int_Z \omega \neq 0$.*

Proof. G. Carron and E. Pedon proved that the map $H_c^1(\mathbb{H}^n/C', \mathbb{R}) \hookrightarrow H_{L^2}^1(\mathbb{H}^n/C', \mathbb{R})$ is injective. The above lemma follows then directly from the duality isomorphism between $H_{n-1}(\mathbb{H}^n/C', \mathbb{R})$ and $H_c^1(\mathbb{H}^n/C', \mathbb{R})$, and from the isometry induced by the Hodge star operator between $H_{L^2}^1$ and $H_{L^2}^{n-1}$. □

In order to find x_0 such that $\mathrm{Jac}_{n-1} F(x_0) = 1$, let us consider, for the harmonic $(n-1)$-form ω of Lemma 1.5,

$$0 < \left|\int_Z \omega\right| = \left|\int_Z \big(F^k\big)^*\omega\right| \leqslant \int_Z \big|\mathrm{Jac}_{n-1} F\big(F^{k-1}(x)\big)\big| \cdots \big|\mathrm{Jac}\, F(x)\big|\, |\omega_{(F^k(x))}|\, \mathrm{d}\,\mathrm{vol}_Z .$$

Here we used that F, and hence F^k, is homotopic to the identity on $\mathbb{H}^n/C'$.

Now let $B = \{x \in Z \mid 0 < \prod_{p=0}^{\infty} \mathrm{Jac}_{n-1} F(F^p(x)) \leqslant 1\}$. The above inequality shows that B has a positive measure.

Assume that for all $x \in B$ the sequence $F^p(x)$ tends to infinity in $\mathbb{H}^n/C'$, then,

$$0 < \left|\int_Z \omega\right| \leqslant \int_B \left(\prod_{p=0}^{k-1} \big|\mathrm{Jac}_{n-1} F\big(F^p(x)\big)\big|\right) |\omega_{(F^k(x))}|\, \mathrm{d}\,\mathrm{vol}_Z + \int_{Z\smallsetminus B} \left(\prod_{p=0}^{k-1} \big|\mathrm{Jac}_{n-1} F\big(F^p(x)\big)\big|\right) |\omega_{(F^k(x))}|\, \mathrm{d}\,\mathrm{vol}_Z .$$

The first integral tends to zero because the product of the Jacobian is bounded above by 1 and

$$\left|\omega\left(F^k(x)\right)\right| \underset{F^k(x)\to\infty}{\longrightarrow} 0.$$

The second integral tends to zero because $|\omega(F^k(x))|$ is bounded and the product of the Jacobians tends to zero by definition of B. This gives a contradiction.

Thus, there exists a point $x \in B$ such that the sequence $F^p(x)$ is bounded in $\mathbb{H}^n/C'$; a subsequence converges to a point x_0 where $\mathrm{Jac}_{n-1} F(x_0) = 1$. □

Step 5: Now $\widetilde{F}$ sends $\mathbb{H}^n$ into the convex hull H of $E(\infty)$. Moreover $D\widetilde{F}(x_0)_{|E} = \mathrm{id}_{|E}$, thus, by the implicit function theorem applied to $\widetilde{F}_{|H}$, there exist open neighbourhoods U and V of x_0 in H such that $\widetilde{F}_{|H}$ sends diffeomorphically U onto V. By equivariance of $\widetilde{F}$, $\widetilde{F}(\gamma U) = \gamma V \subset H$ for all $\gamma \in C$. Thus, if $\gamma \in C$, γ preserves H. Now $H_{n-1}(C', \mathbb{R}) \simeq H_{n-1}(H/C, \mathbb{R})$ and is nonzero which implies that $Y = H/C$ is compact. In this case $\widetilde{F}$ is the orthogonal projection onto H and Y is homotopic to Z hence it separates X. □

In fact the proof sketched above shows the following

Theorem 1.6. *Let C be a discrete subgroup of isometries of $\mathbb{H}^n$ acting without fixed point such that there exists an essential p-dimensional compact submanifold $Z \subset \mathbb{H}^n/C$. Assume furthermore that the injectivity radius of $\mathbb{H}^n/C$ is positive and $p \geqslant 3$, then*

$$\delta(C) \geqslant p - 1.$$

One advantage of the proof sketched above is its flexibility. More precisely the same proof gives a similar result for X a manifold with sectional curvature less than -1. The details will be published in a forthcoming paper. In this article we shall also give the extension of the above result (in the variable curvature case) to the case where Γ is a graph of groups. The complex hyperbolic case will be treated too.

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