Available online at www.sciencedirect.com
SCIENCE DIRECT
C. R. Acad. Sci. Paris, Ser. I 336 (2003) 85–88
COMPTES RENDUS MATHEMATIQUE

Statistics/Probability Theory

Tail behavior of anisotropic norms for Gaussian random fields

Comportement des queues pour les normes anisotropes des champs aléatoires gaussiens

Mikhail Lifshits, Alexander Nazarov, Yakov Nikitin

Department of Mathematics and Mechanics, St. Petersburg State University, St. Petersburg 198504, Russia

Received 20 March 2002; accepted after revision 13 November 2002

Presented by Paul Deheuvels

Abstract

We investigate the logarithmic large deviation asymptotics for anisotropic norms of Gaussian random functions of two variables. The problem is solved by the evaluation of the anisotropic norms of corresponding integral covariance operators. We find the exact values of such norms for some important classes of Gaussian fields. ***To cite this article: M. Lifshits et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous étudions les grandes déviations logarithmiques pour les normes anisotropes des champs gaussiens aléatoires de deux variables. Le problème est résolu en calculant des normes anisotropes pour les opérateurs intégraux engendrés par les covariances. Nous trouvons des valeurs exactes de telles normes pour quelques classes importantes de champs gaussiens. ***Pour citer cet article : M. Lifshits et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. We start with some classical examples which motivated our research. Consider three Gaussian fields on the unit square I^2: the Brownian sheet $W_1(x_1,x_2)$ with the covariance $\mathcal{K}_1(x_1,x_2;y_1,y_2)=(x_1\wedge y_1)(x_2\wedge y_2)$, the Brownian pillow $W_2(x_1,x_2)$ with the covariance $\mathcal{K}_2(x_1,x_2;y_1,y_2)=(x_1\wedge y_1-x_1y_1)(x_2\wedge y_2-x_2y_2)$, and the Kiefer field $W_3(x_1,x_2)$ with the covariance $\mathcal{K}_3(x_1,x_2;y_1,y_2)=(x_1\wedge y_1)(x_2\wedge y_2-x_2y_2)$.

For any field ξ on I^2 and for $1\leqslant p_1,p_2\leqslant\infty$ introduce the anisotropic norm

$$\|\xi\|_{p_1,p_2}=\left[\int_0^1\left[\int_0^1\left|\xi(x_1,x_2)\right|^{p_1}\mathrm{d}x_1\right]^{p_2/p_1}\mathrm{d}x_2\right]^{1/p_2}. \qquad (1)$$

E-mail addresses: lifts@mail.rcom.ru (M. Lifshits), an@an4751.spb.edu (A. Nazarov), Yakov.Nikitin@pobox.spbu.ru (Ya. Nikitin).

1631-073X/03/$ – see front matter
doi:10.1016/S1631-073X(02)00013-4

We are interested in the logarithmic tail behavior of this norm for our three fields, in other words, we look for the constants $C_i = \lim_{z\to\infty} z^{-2} \ln P\{\|W_i\|_{p_1,p_2} \geqslant z\},\ i = 1,2,3$.

Such constants are important, in particular, when calculating the Bahadur efficiency of nonparametric integral tests of independence, see [9]. The following solution is a special case of the results in [8]. Let

$$\sigma(p) = \frac{2}{p\pi}\left(1+\frac{p}{2}\right)^{(p-2)/p}\left(\frac{\Gamma(\frac{1}{2}+\frac{1}{p})}{\Gamma(1+\frac{1}{p})}\right)^2, \quad 1 \leqslant p.$$

It was proved in [8] that

$$\lim_{z\to\infty} z^{-2} \ln P\bigl\{\|W_1\|_{p_1,p_2} \geqslant z\bigr\} = -\bigl(2\sigma(p_1)\sigma(p_2)\bigr)^{-1},$$
$$\lim_{z\to\infty} z^{-2} \ln P\bigl\{\|W_2\|_{p_1,p_2} \geqslant z\bigr\} = -8\bigl(\sigma(p_1)\sigma(p_2)\bigr)^{-1},$$
$$\lim_{z\to\infty} z^{-2} \ln P\bigl\{\|W_3\|_{p_1,p_2} \geqslant z\bigr\} = -2\bigl(\sigma(p_1)\sigma(p_2)\bigr)^{-1}.$$

The aim of this Note is to obtain such large deviation results for a larger class of Gaussian fields.

2. First, we recall a general approach to the calculation of large deviation constants. Let $T_j,\ j = 1,2$, be two measure spaces and let $X(t_1,t_2),\ \ t_j \in T_j$, be a Gaussian random function on the set $\mathcal{T} = T_1 \times T_2$ with the covariance function $\mathcal{K}(s_1,s_2;t_1,t_2)$ on $\mathcal{T}\times\mathcal{T}$. Let $L_{p_1,p_2}(\mathcal{T})$ be the anisotropic Lebesgue space with the norm (1), $1 \leqslant p_1, p_2 \leqslant \infty$. Consider the integral operator $\mathcal{K}: L_{r_1,r_2}(\mathcal{T}) \to L_{p_1,p_2}(\mathcal{T})$ defined by the formula $(\mathcal{K}g)(s_1,s_2) = \int_{\mathcal{T}} \mathcal{K}(s_1,s_2;t_1,t_2)g(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$ with the norm $\|\mathcal{K}\|_{(r_1,r_2)\to(p_1,p_2)}$. Let $q_j,\ j = 1,2$, be the dual exponents to p_j, i.e., $p_j^{-1} + q_j^{-1} = 1$. It is known, see [6, Section 12], that the constant $\|\mathcal{K}\|_{(q_1,q_2)\to(p_1,p_2)}$ describes the logarithmic large deviation asymptotics of X in the space $L_{p_1,p_2}(\mathcal{T})$, namely,

$$\lim_{z\to\infty} z^{-2} \ln P\bigl\{\|X\|_{p_1,p_2} \geqslant z\bigr\} = \bigl(2\|\mathcal{K}\|_{(q_1,q_2)\to(p_1,p_2)}\bigr)^{-1}. \tag{2}$$

We want to calculate $\|\mathcal{K}\|_{(q_1,q_2)\to(p_1,p_2)}$ for as large class of fields X as possible. For each of three classical fields described above we have the remarkable property

$$\mathcal{K}(s_1,s_2;t_1,t_2) = K_1(s_1,t_1)\cdot K_2(s_2,t_2) \tag{3}$$

for some covariances $K_1(\cdot,\cdot)$ and $K_2(\cdot,\cdot)$. We wonder if the result on large deviations of the anisotropic norms for fields W_1–W_3 can be generalized to the class of Gaussian fields satisfying the property (3). Towards this aim, let us slightly reformulate our problem in terms of the theory of tensor products.

The tensor product of integral operators with the kernels K_1 and K_2 is the integral operator with the kernel $K_1 \otimes K_2((s_1,s_2),(t_1,t_2)) = K_1(s_1,t_1)\cdot K_2(s_2,t_2)$ acting from $L_{r_1,r_2}(\mathcal{T})$ into $L_{p_1,p_2}(\mathcal{T})$ according to the formula $((K_1\otimes K_2)g)(s_1,s_2) = \int_{\mathcal{T}} K_1(s_1,t_1)\cdot K_2(s_2,t_2)g(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$.

Theorem 1. *Consider two integral operators* $K_j : L_{r_j}(T_j) \to L_{p_j}(T_j),\ j = 1,2$, *with the norms* $\|K_j\|_{r_j\to p_j}$. *Let one of the following conditions be fulfilled*: (1) $K_2 \geqslant 0$; (2) $p_2 \geqslant p_1 \geqslant r_2$; (3) $p_2 \geqslant r_1 \geqslant r_2$. *Then*

$$\|K_1 \otimes K_2\|_{(r_1,r_2)\to(p_1,p_2)} = \|K_1\|_{r_1\to p_1}\cdot\|K_2\|_{r_2\to p_2}. \tag{4}$$

Remark. In the isotropic case ($p_1 = p_2, r_1 = r_2$) the functional analysts have extensively investigated whether the equality

$$\|K_1 \otimes K_2\|_{r\to p} = \|K_1\|_{r\to p}\cdot\|K_2\|_{r\to p} \tag{5}$$

is true. Bennett [1] was apparently the first who proved (for operators acting from l_r to l_p) that (5) is true for any K_1 and K_2 iff $1 \leqslant r \leqslant p \leqslant \infty$. Later his result was rediscovered and generalized, see, e.g., [10]. Therefore, (4) cannot be true without additional assumptions.

Let us simplify the large deviation result (2) for the anisotropic norm of Gaussian fields having the covariance structure (3). In this case $r_j = q_j$, so that the conditions (2) and (3) of Theorem 1 are equivalent. Thus we obtain the next theorem.

Theorem 2. *Assume that a Gaussian field has the covariance* (3), *and let one of the following conditions be fulfilled*: (1) $K_2 \geqslant 0$; (2) $p_2 \geqslant p_1 \geqslant q_2$. *Then the following logarithmic large deviation asymptotics holds*:

$$\lim_{z\to\infty} z^{-2} \ln P\big\{\|X\|_{p_1,p_2} \geqslant z\big\} = -\big(2\|K_1\|_{q_1\to p_1} \cdot \|K_2\|_{q_2\to p_2}\big)^{-1}. \tag{6}$$

Using this result, we can reduce the calculation of "two-dimensional" norm to that of "one-dimensional" ones which may be already known or may be calculated independently in a much easier way. The classical Brownian fields W_1–W_3 have nonnegative covariances and satisfy the conditions of Theorem 2. Hence the results of [8] stated above follow from Theorem 2. On the other hand, there are important examples of Gaussian fields which satisfy (3) but have sign-alternating covariances. In this case we can establish the asymptotic behavior (6) only under some additional assumptions on p_j, $j = 1, 2$.

We mention the following generalization of Theorem 1 in the spirit of the theory of tensor products. It concerns more general operators and more general norms in functional spaces.

Theorem 3. *Let* $1 \leqslant r \leqslant p$. *Consider continuous linear operators* $K_1 : X_1 \to Y_1$, $K_2 : X_2 \to Y_2$, *where* X_1, Y_2 *are arbitrary Banach spaces while* $Y_1 = L_p(T_1)$ *and* $X_2 = L_r(T_2)$. *Then the tensor product* $K_1 \otimes K_2 : X_1 \otimes X_2 = L_r(T_2; X_1) \to Y_1 \otimes Y_2 = L_p(T_1; Y_2)$ *is continuous and* $\|K_1 \otimes K_2\| \leqslant \|K_1\| \cdot \|K_2\|$.

Parts (2) and (3) of Theorem 1 can be deduced from this result.

3.

Example 1 (*Rothmann field*). When constructing the tests of independence on a torus, Rothmann introduced in [11] the Gaussian field $\rho(x_1, x_2)$ with zero mean and the covariance function $\mathcal{C}(x_1, x_2; y_1, y_2) = C(x_1, y_1)C(x_2, y_2)$ with the "marginal" kernel $C(x, y) = (x \wedge y - \frac{1}{2}(x + y) + \frac{1}{2}(x - y)^2 + \frac{1}{12})$. The covariance of this field satisfies (3) but is not nonnegative. Denote by $\mathcal{C}$ the integral operator with the kernel $\mathcal{C}(x_1, x_2; y_1, y_2)$ and by C the integral operator with the kernel $C(x, y)$. Applying Part 2 of Theorem 2 we obtain that for $2 \leqslant p_2 \leqslant \infty$, $q_2 \leqslant p_1 \leqslant p_2$

$$\|\mathcal{C}\|_{(q_1,q_2)\to(p_1,p_2)} = \|C \otimes C\|_{(q_1,q_2)\to(p_1,p_2)} = \|C\|_{q_1\to p_1} \cdot \|C\|_{q_2\to p_2}.$$

We just have to evaluate the constants $\tau(p) := \|C\|_{q\to p}$ which may be treated in terms of the extremal problem

$$\frac{1}{(\tau(p))^{1/2}} = \min \frac{\|u'\|_2}{\|u - \int_0^1 u(x)\,\mathrm{d}x\|_p}, \quad u \geqslant 0,\ u(0) = u(1) = 0.$$

It was proved in [7] (see there the history of the problem) that for $1 \leqslant p \leqslant 6$ one has $\tau(p) = \sigma(p)/16$, while for $p > 6$ it holds [2] $\tau(p) > \sigma(p)/16$. Thus at least for $2 \leqslant p_2 \leqslant 6$, $q_2 \leqslant p_1 \leqslant p_2$ we derive from (6) that $\lim_{z\to\infty} z^{-2} \ln P\{\|\rho\|_{p_1,p_2} \geqslant z\} = -128(\sigma(p_1)\sigma(p_2))^{-1}$.

Example 2 (*Integrated Brownian sheet, pillow and related fields*). Let $W_2(x_1, x_2)$ be the Brownian pillow on the unit square.

Consider the integrated Brownian pillow $Y_2(s_1, s_2) = \int_0^{s_1}\int_0^{s_2} W_2(x_1, x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1$ having the covariance function $\mathcal{D}_2(s_1, s_2; t_1, t_2) = D_2(s_1, t_1)D_2(s_2, t_2)$, where $D_2(s, t) = \frac{1}{2} st(s \wedge t) - \frac{1}{6}(s \wedge t)^3 - \frac{1}{4} s^2t^2 \geqslant 0$. We can apply Part 1 of Theorem 2, whenever we know the values of corresponding norms. Consider the Hilbert case $p_1 = p_2 = 2$. The spectrum of the integral operator with the kernel D_2 was found in [3]. Let $\mu_1 < \mu_2 < \cdots$ be the solutions of the equation $\tan(\mu) + \tanh(\mu) = 0$. Then $\lambda_n = (\mu_n)^{-4}$, $n \geqslant 1$. In particular, $\lambda_1 \approx 3.20 \times 10^{-2}$. On the other hand, it is well known that the norm of symmetric operator in Hilbert space is equal to its first eigenvalue [4, Chapter V]. Hence in the case $p_1 = p_2 = 2$ we obtain $\lim_{z\to\infty} z^{-2} \ln P\{\|Y_2\|_2 \geqslant z\} = -(2\lambda_1^2)^{-1} \approx -489.39$.

Another case covered by our theorems is the case when one of the exponents equals 2, and the another equals infinity. The value of the norm of the integral operator acting from $L_1(T)$ into $L_\infty(T)$ is also well-known (see again [4, Chapter V]), namely, $\|D_2\|_{1\to\infty} = \sup_{s,t} D_2(s, t) = 1/12$, so that we get

$$\lim_{z\to\infty} z^{-2}\ln P\bigl\{\|Y_2\|_{2,\infty}\geqslant z\bigr\} = \lim_{z\to\infty} z^{-2}\ln P\bigl\{\|Y_2\|_{\infty,2}\geqslant z\bigr\} = -6\lambda_1^{-1}\approx -187.71. \tag{7}$$

We may consider similarly the integrated Wiener sheet $Y_1(s_1,s_2)=\int_0^{s_1}\int_0^{s_2} W_1(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$ which is the Gaussian field with the covariance $\mathcal{D}_1(s_1,s_2;t_1,t_2)=D_1(s_1,t_1)\cdot D_1(s_2,t_2)$, where $D_1(s,t)=\frac{1}{2}st(s\wedge t)-\frac{1}{6}(s\wedge t)^3\geqslant 0$.

Arguing as in [3] we see that the spectrum of the integral operator with the kernel D_1 consists of eigenvalues $\tilde\lambda_n=\nu_n^{-4}$, $n\geqslant 1$, where $\nu_1<\nu_2<\cdots$ are the solutions of the equation $\cos(\nu)\cosh(\nu)+1=0$. It follows that the first eigenvalue is $\tilde\lambda_1\approx 8.09\cdot 10^{-2}$. Hence

$$\lim_{z\to\infty} z^{-2}\ln P\bigl\{\|Y_1\|_2\geqslant z\bigr\} = -\bigl(2\tilde\lambda_1^2\bigr)^{-1}\approx -76.42. \tag{8}$$

One can treat other integrated fields including the integrated Kiefer field in the same way getting the results analogous to (7) and (8).

Example 3 (*Ornstein–Uhlenbeck sheet*). Consider the Ornstein–Uhlenbeck sheet on I^2 which can be defined as a Gaussian field $U(s_1,s_2)$ with zero mean and covariance $\mathcal{U}(s_1,s_2;t_1,t_2)=\exp(-|s_1-t_1|-|s_2-t_2|)$. This covariance fits Part 1 of Theorem 2. We must find the norms of the corresponding univariate integral operators acting from L_q into L_p. Actually we have the solution only for $p=2$. We need the first eigenvalue of the integral equation $\lambda f(t)=\int_0^1\exp(-|s-t|)f(s)\,\mathrm{d}s$. The spectrum of this equation was found in [5]. It consists of eigenvalues $\hat\lambda_n=2(\eta_n^2+1)^{-1}$, $n\geqslant 1$, where $\eta_1<\eta_2<\cdots$ are the roots of the equation $2\eta\cos(\eta)=(\eta^2-1)\cdot\sin(\eta)$. Solving this equation numerically we find the first eigenvalue $\hat\lambda_1\approx 0.74$. It follows that $\lim_{z\to\infty} z^{-2}\ln P\{\|U\|_2\geqslant z\}=-(2\hat\lambda_1^2)^{-1}\approx -0.92$.

Example 4 (*Fractional Brownian sheet*). Consider a Gaussian field with the zero mean and covariance

$$B_\alpha(s_1,s_2;t_1,t_2)=\beta(s_1,t_1)\cdot\beta(s_2,t_2),\quad 0<\alpha<2, \tag{9}$$

where $\beta(s,t)=\frac{1}{2}(s^\alpha+t^\alpha-|s-t|^\alpha)$.

The univariate kernels in (9) are nonnegative, thus Part 1 of Theorem 2 works. However, the norms and eigenvalues for the operators with such kernels are unknown for $\alpha\neq 1$.

Acknowledgements

This work was supported by grants of RFBR No.00-15-96019 and 01-01-00245 and by the grant of Russian Ministry of Education No. E00-1-124.

References

[1] G. Bennett, Schur multipliers, Duke Math. J. 44 (1977) 603–639.
[2] A.P. Buslaev, V.A. Kondrat'ev, A.I. Nazarov, On a family of extremum problems and the properties of an integral, Math. Notes 64 (1998) 719–725.
[3] N. Henze, Ya.Yu. Nikitin, A new approach to goodness-of-fit testing based on the integrated empirical process, J. Nonparametr. Statist. 12 (2000) 391–416.
[4] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, 1982.
[5] W.V. Li, Comparison results for the lower tail of Gaussian seminorms, J. Theoret. Probab. 5 (1992) 1–31.
[6] M.A. Lifshits, Gaussian Random Functions, Kluwer Academic Publishers, 1995.
[7] A.I. Nazarov, On exact constant in the generalized Poincaré inequality, Probl. Math. Anal. 24 (2002) 155–180 (in Russian). J. Math. Sci. 112 (2002) 4029–4047.
[8] A.I. Nazarov, Ya.Yu. Nikitin, Some extremal problems for Gaussian and empirical random fields, Proc. St. Petersburg Math. Soc. 8 (2000) 214–230 (in Russian). Amer. Math. Soc. Trans. Ser. 2 205 (2002) 189–202.
[9] Ya.Yu. Nikitin, Asymptotic Efficiency of Nonparametric Tests, Cambridge University Press, 1995.
[10] H.P. Rosenthal, S.J. Sharek, On tensor products of operators from L_p to L_q, in: Lecture Notes in Math., Vol. 1470, Springer, 1991, pp. 108–132.
[11] E.D. Rothmann, Tests of coordinate independence for a bivariate sample on a torus, Ann. Math. Statist. 42 (1971) 1962–1969.