# PERCOLATION OF RANDOM NODAL LINES

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ABSTRACT

We prove a Russo-Seymour-Welsh percolation theorem for nodal domains and nodal lines associated to a natural infinite dimensional space of real analytic functions on the real plane. More precisely, let U be a smooth connected bounded open set in $\mathbf{R}^2$ and $\gamma, \gamma'$ two disjoint arcs of positive length in the boundary of U. We prove that there exists a positive constant $c$, such that for any positive scale $s$, with probability at least $c$ there exists a connected component of the set $\{x \in \bar{\mathrm{U}},\ f(sx) > 0\}$ intersecting both $\gamma$ and $\gamma'$, where $f$ is a random analytic function in the Wiener space associated to the real Bargmann-Fock space. For $s$ large enough, the same conclusion holds for the zero set $\{x \in \bar{\mathrm{U}},\ f(sx) = 0\}$. As an important intermediate result, we prove that sign percolation for a general stationary Gaussian field can be made equivalent to a correlated percolation model on a lattice.

CONTENTS



## 1. Introduction

In this paper, we prove that for a natural infinite dimensional space of analytic functions, for any fixed open connected set U in the real plane, with uniformly positive probability there exist connected components of the zero set of the random function which cross arbitrary large homothetical copies of U. This result lies at the intersection of two almost disjoint fields of geometric probability. The first one involves the geometry of zeros of smooth Gaussian functions on the affine real space or on a compact manifold, the other concerns percolation, Ising model and Gaussian free field for instance. A main qualitative difference between the two fields is the presence in the second one of large

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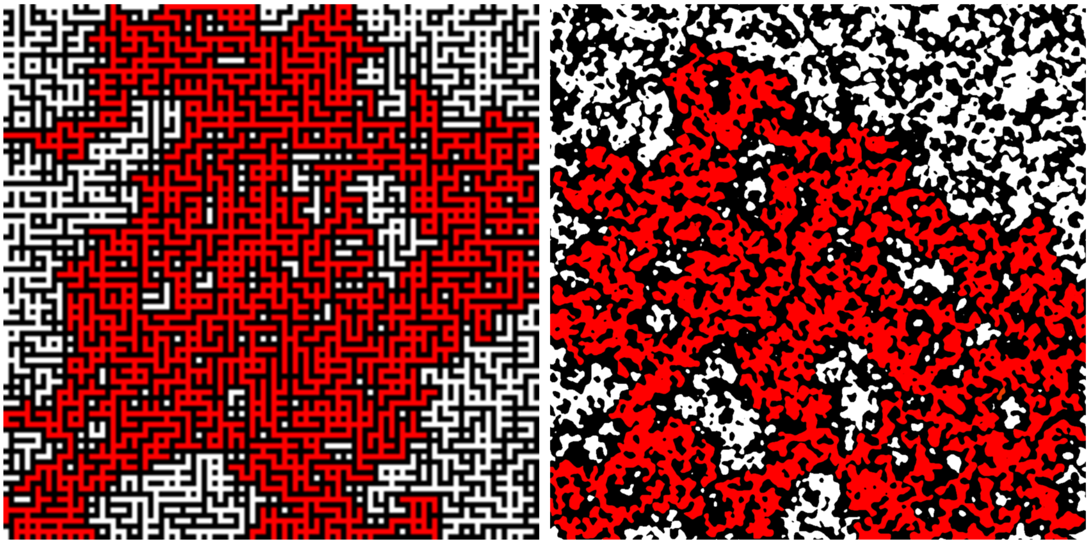

FIG. 1. — *Left*: critical Bernoulli percolation. In *red*, a percolating cluster. *Right*: analytic percolation. In *red*, a percolating nodal domain (color figure online)

scale phenomena arising from the microscopic behavior, beginning with Russo-Seymour-Welsh theory in percolation, see below. Our result establishes large scale phenomena happening in the first field, using methods coming from both sides.

In this introduction, we first recall some topics and results in these classical domains, then we state our main results, we discuss how the Bargmann-Fock space is related to random algebraic geometry, we recall the Bogomolny-Schmidt conjecture and lastly we propose some open questions.

*Geometry and topology of random nodal sets.* — For a stationary (*i.e.* invariant in distribution under translations) Gaussian field $f$ on the affine space $\mathbf{R}^n$, two main subjects have been studied, namely the statistics of the volume and the Euler characteristic of zero sets of $f$ in a large ball, see for instance the book [1] and references therein. Since these observables can be computed locally by integral geometry and the Gaussian field is smooth, the main tool is the Kac-Rice formula.

On a compact manifold, two very natural types of Gaussian random functions have been studied. If M is equipped with a Riemannian metric, we can consider Gaussian random independent sums of eigenfunctions of the Laplacian with eigenvalues less than a parameter L. If M denotes the complex or real projective space, one can study random homogeneous polynomials of degree $d$ with Gaussian independent coefficients (more generally holomorphic sections of the $d$-th power of a holomorphic line bundle over a Kähler manifold or its real part). For nodal sets of positive dimension, their mean volume was first studied in [10] (in the Riemannian case). In [41], the authors studied the integration current over nodal hypersurfaces in a very general complex algebraic setting. Note that in this case, and contrary to the real algebraic case, the topology of the random set is deterministic. In [38] the mean Euler characteristic was computed (in the real algebraic case).

In [34] and [20], observables which cannot be computed locally were estimated, beginning with the number of components the zero set of the random function. In [23], higher Betti numbers were studied. The universal presence of given diffeomorphic types (included arrangements of ovals) in the zero sets was proved in [22]. The survey [4] provides further references.

In both models, the correlation function of the Gaussian field rescales naturally under the action of the parameter (the eigenvalue bound L or the degree $d$) and converges to a universal kernel on the affine space. Finding the limit of this rescaling is trivial for standard geometries like the round sphere or the flat torus, and in general can be extracted from deep results of semi-classical analysis [29] and complex analysis [45, 46]. In the general algebraic case, this kernel is precisely the kernel we use in this paper, see [11]. Hence, the Bargmann-Fock model is a natural universal algebraic limit model.

*Percolation.* — The main contribution of this paper is to bring to the topic of random nodal lines ideas and techniques originated from percolation theory. In its simplest form, Bernoulli percolation [16] is defined as follows: color each vertex of a periodic lattice $\mathcal{T}$ independently either black or white, with probability $p$ and $1-p$ respectively, and define the random subgraph G of $\mathcal{T}$ formed of all the vertices colored black and of the edges joining them. There exists a critical parameter $p_c$ such that G has a.s. no infinite connected component if $p < p_c$, and a.s. at least one infinite component if $p > p_c$; what will be relevant to us is the behavior of G at the critical point, in dimension 2. We refer the reader to [7, 26] for further references on the model.

If $\mathcal{T}$ is a periodic triangulation of the plane with enough symmetry (in practice, one typically works on the triangular lattice or the "Union-Jack lattice", *i.e.* the face-centered square lattice), it is a classical result tracing back to Kesten [30] that $p_c = 1/2$; the fundamental reason for this being a *duality* between white and black clusters on $\mathcal{T}$, each finite cluster being surrounded by a cluster of the other color. It was proved by Harris [28] that at the critical point, with probability 1 the random graph G has no infinite component.

One crucial technique on critical two-dimensional percolation, which we will extend to the setup of random nodal lines, is known as Russo-Seymour-Welsh theory [39, 40] and leads to the *box-crossing property* of Bernoulli percolation: namely, for every $\rho > 1$ there exists a positive bound $c(\rho)$ such that every rectangle of size $\rho s \times s$ is traversed in its long dimension by a black cluster with probability at least $c(\rho)$, uniformly in $s$. This readily extends to the crossing of *quads*, which are regions of the form $s$U where U is a simply connected domain with two disjoint marked boundary intervals $\gamma$ and $\gamma'$: the probability that $s$U contains a black cluster connecting $s\gamma$ to $s\gamma'$ is bounded below uniformly in $s$.

The box-crossing property has many consequences. First, it implies that even though there is no infinite cluster, every box of size $n$ contains a black cluster of diameter of order $n$. It also implies the non-existence of an infinite cluster and moreover quantifies it: the probability that the origin is in a cluster of diameter L is then bounded above by $L^{-\eta}$ for some $\eta > 0$. It is the main tool used in the control of the geometry of critical clusters [2], and a fundamental ingredient in the obtention of Schramm-Loewner Evolution as a scaling limit [17, 43].

Russo-Seymour-Welsh theory has been extended to dependent models [18], as well as to some random [15, 44] or non-planar [6, 36] lattices. It remains an open problem to generalize it to triangulations without symmetries. Our main statement in this paper is that the box-crossing property applies to random nodal lines.

*The main result.* — Let $\mathcal{A}$ be the space of real analytic functions on $\mathbf{R}^2$, and $\langle , \rangle_{\mathrm{BF}}$ the scalar product defined, when it exists, by

$$\forall f, g \in \mathcal{A}, \quad \langle f, g\rangle_{\mathrm{BF}} = \int_{\mathbf{C}^2} \mathrm{F}(z)\overline{\mathrm{G}(z)} e^{-\|z\|^2} \frac{dx}{\pi^2}, \tag{1.1}$$

where F and G are the complex extensions of $f$ and $g$. The real Bargmann-Fock space $\mathcal{F}$ is the space of functions in $\mathcal{A}$ which have finite norm for this product. This Hilbert space induces a natural abstract Wiener space $\mathcal{W}(\mathcal{F})$ of analytic functions, see Appendix A.1. More concretely, we can choose a Hilbert basis of $\mathcal{F}$, for instance the monomials $(\frac{1}{\sqrt{i!j!}} x_1^i x_2^j)_{i,j\in\mathbf{N}}$. Then, a random function of $\mathcal{W}(\mathcal{F})$ is of the form

$$\forall x = (x_1, x_2) \in \mathbf{R}^2, \quad f(x) = \sum_{i,j=0}^{\infty} a_{ij} \frac{1}{\sqrt{i!j!}} x_1^i x_2^j, \tag{1.2}$$

where the coefficients $(a_{ij})_{i,j\in\mathbf{N}}$ are independent real normal variables. For almost all realizations of the coefficients, this sum does not converge in $\mathcal{F}$, but almost surely it converges uniformly on every compact of $\mathbf{R}^2$, so that it converges as an analytic function. Note that the correlation function of this Gaussian field equals

$$\forall (x,y) \in \left(\mathbf{R}^2\right)^2, \quad e_{\mathcal{W}(\mathcal{F})}(x,y) = \mathbf{E}_{\mathcal{W}(\mathcal{F})}\big(f(x)f(y)\big) = \exp\langle x, y\rangle.$$

A more convenient way to see this set of random functions is to choose, as our space of random analytic functions,

$$\mathcal{W} = \left\{ x \mapsto f(x) \exp\left(-\frac{1}{2}\|x\|^2\right), \, f \in \mathcal{W}(\mathcal{F}) \right\} \tag{1.3}$$

instead of $\mathcal{W}(\mathcal{F})$. Notice that the signs and the zeros of the functions are not changed when passing from one space to the other. Then, it is immediate to see that the correlation function associated to $\mathcal{W}$ equals

$$\forall (x,y) \in \left(\mathbf{R}^2\right)^2, \quad e_{\mathcal{W}}(x,y) = \mathbf{E}_{\mathcal{W}}\big(g(x)g(y)\big) = \exp\left(-\frac{1}{2}\|x-y\|^2\right). \tag{1.4}$$

In other words, this model is in fact the analytic stationary Gaussian field on $\mathbf{R}^2$ with covariance $e_{\mathcal{W}}$.

*Theorem* **1.1.** — *Let $\Gamma = (\mathrm{U}, \gamma, \gamma')$ be a quad, that is a triple given by a smooth bounded open connected set $\mathrm{U} \subset \mathbf{R}^2$, and two disjoint compact smooth arcs $\gamma$ and $\gamma'$ in $\partial\mathrm{U}$. Then, there exists a positive constant c such that for any positive s, with probability at least c there exists a connected component of $\{x \in \bar{\mathrm{U}}, f(sx) > 0\}$ intersecting both $\gamma$ and $\gamma'$, where $f$ is a random element of $\mathcal{W}(\mathcal{F})$ or $\mathcal{W}$ with distribution as above. For s large enough, the same conclusion holds for the nodal set $\{x \in \bar{\mathrm{U}}, f(sx) = 0\}$.*

*Remark* **1.2.** — In fact, we prove the more general Theorem 4.9 which holds for $f$ being a $\mathrm{C}^4$ random stationary Gaussian field whose correlation function is positive, is invariant under horizontal symmetry, and decreases polynomially with the distance between points, for a degree larger than $144 + 128 \log_{4/3}(3/2) < 325$. This number should not be taken too seriously, since it can be lowered if the constants in the proofs are more accurately estimated. After we posted a preliminary version of this paper online, Beliaev and Muirhead [9] developed a more precise discretization scheme which allowed them to lower this bound to 16.

*Remark* **1.3.** — Note that the probability that a given $\varepsilon \times \varepsilon$ square intersects a nodal line at all goes to zero with $\varepsilon$ (see Lemma 3.12). In particular, the restriction on $s$ for nodal crossings in Theorem 1.1 is necessary.

The usual consequences of the box-crossing property for percolation include estimates for the probabilities of various connection events, and they all apply here. We do not make a full list but just choose to mention one striking corollary:

*Theorem* **1.4.** — *Let $\pi(s,t)$ be the probability that there exists a positive line (resp. a nodal line) from $[-s,s]^2$ to the boundary of $[-t,t]^2$. There exists $\eta > 0$ and $\mathrm{C} > 0$, such that, for every $1 \leqslant s < t$,*

$$\pi(s,t) \leqslant \mathrm{C}\left(\frac{s}{t}\right)^{\eta}.$$

The following corollary was already proved by K. S. Alexander, see [3], for the nodal lines and for general positive kernels.

*Corollary* **1.5.** — *With probability 1, none of the sets $f^{-1}(\{0\}), f^{-1}(\mathbf{R}_+^*)$ and $f^{-1}(\mathbf{R}_-^*)$ has an unbounded connected component.*

*Some heuristics.* — In Theorem 1.1, the first assertion concerning nodal domains can be seen as the statement of a box-crossing type property for the random coloring of the plane given by the sign of the function $f$. We therefore have to prove that a crossing where $f$

is positive exists with uniformly positive probability, which is precisely the kind of results produced by Russo-Seymour-Welsh theory.

Moreover, the second assertion can be deduced from the first one, if there is enough independence for the sign of $f$ between two disjoint domains. Indeed, the simultaneous existence of a path from $\gamma$ to $\gamma'$ along which $f$ is positive, and a similar path along which it is negative, implies the existence of a component of the zero set of $f$ in between.

The biggest issue to overcome is the rigidity created by the analyticity of the function $f$. We are basing our approach on a RSW result for dependent models by Tassion [44], but even though the correlation kernel of $f$ has fast decay, this does not imply that the restrictions of $f$ to disjoint open sets are decorrelated—to the contrary, analytic continuation shows that the restriction of $f$ to any neighborhood specifies $f$ in the whole plane. Even knowing the sign of $f$ on an open set V can be enough to reconstruct $f$ up to a multiplicative constant, as soon as V intersects the zero set of $f$.

To address the issue, we will discretize the model by considering the restriction of $f$ to the vertices of a lattice of mesh $\varepsilon$ lying in our rescaled domain $s\mathrm{U}$. The mesh has to be chosen with some care: it has to be coarse enough for the restricted model to have small correlation, and to avoid the effects of the rigidity; but at the same time it should be fine enough to ensure that the knowledge of the sign of $f$ on the lattice suffices to determine the topology of its nodal domains.

Note that in [33], the authors also used a discretization procedure to show the existence of a phase transition for certain continuous fields. However, since they work far away from criticality, the comparison arguments are technically much simpler.

*Discretization.* — One of our main tools is Theorem 1.6, which holds for general smooth enough stationary Gaussian fields, hence has its own interest.

*Theorem* **1.6.** — *Let $f$ be a $\mathrm{C}^4$ random stationary Gaussian field on $\mathbf{R}^2$ satisfying the non-degeneracy condition (3.5), $\mathcal{T}$ be a periodic lattice, and $\mathcal{E}$ be its set of edges. Fix $\eta > 0$. Then there exists $s(\eta) > 0$, $\mathrm{C}(\eta) > 0$ and $\alpha(\eta) > 0$ such that for every $s \geqslant s(\eta)$, every $\delta > 0$ and every $\varepsilon \leqslant \mathrm{C}(\eta)\delta^{\alpha(\eta)}s^{-8-\eta}$, with probability at least $1-\delta$ the following event happens:*

$$\forall e \in \varepsilon\mathcal{E} \cap \mathrm{B}_s, \quad \#\big(e \cap f^{-1}(0)\big) \leqslant 1. \tag{1.5}$$

This theorem proves that for every scale $s$ large enough, the discretization on a lattice of mesh small enough (polynomially decreasing with $s$), provides on the box $[-s,s]^2$ a family of percolation processes which catch the topology of the continuous percolation process. Indeed, under the event (1.5), two adjacent vertices in $\varepsilon\mathcal{T}$ are of same sign for $f$ if and only if $f$ is of this constant sign on the edge between them, so that a percolation for positive signs in $\varepsilon\mathcal{T}$ through the length of any rectangle inside $[-s,s]^2$ means exactly the existence of an analogous continuous positive percolation for the field.

*Bargmann-Fock and polynomials.* — Consider a real random homogeneous polynomial of degree $d$,

$$(\mathbf{1.6}) \qquad \mathrm{P}(\mathrm{X}) = \sum_{|\mathrm{I}|=d} a_{\mathrm{I}} \sqrt{\frac{(d+n)!}{\mathrm{I}!}} \mathrm{X}^{\mathrm{I}},$$

where $\mathrm{X} = (\mathrm{X}_0, \dots, \mathrm{X}_n) \in \mathbf{R}\mathrm{P}^n$ are the projective coordinates, $\mathrm{I} = (i_0, \dots, i_n) \in \mathbf{N}^{n+1}$, $|\mathrm{I}| = i_0 + \cdots + i_n$, $\mathrm{I}! = i_0! \cdots i_n!$, $\mathrm{X}^{\mathrm{I}} = \mathrm{X}_0^{i_0} \cdots \mathrm{X}_n^{i_n}$ and the $(a_{\mathrm{I}})_{\mathrm{I}}$ are independent normal random coefficients. This measure on polynomials is very natural because it has a geometric nature: the monomials form an orthonormal basis for the Fubini-Study $\mathrm{L}^2$-norm

$$\|\mathrm{P}\|^2 = \int_{\mathbf{C}\mathrm{P}^n} \|\mathrm{P}(z)\|^2_{\mathrm{FS}} dvol(z),$$

where the $\|\cdot\|_{\mathrm{FS}}$ denotes the Fubini-Study norm on the $d$-th power $\mathcal{O}(d)$ of the hyperplane bundle over the complex projective space, and *dvol* is the uniform volume form on $\mathbf{C}\mathrm{P}^n$ of total volume equal to 1. Note that we integrate over the complex manifold $\mathbf{C}\mathrm{P}^n$ and not $\mathbf{R}\mathrm{P}^n$. Kostlan [31] and independently Shub and Smale [42] proved, for instance, that with this measure and for $n = 1$, the average number of real roots of a real polynomial of degree $d$ is precisely $\sqrt{d}$, for every $d$. This algebraic random model is in fact very general and holds in the realm of holomorphic sections of the high powers $d$ of an ample line bundle over any compact complex manifold (see [41]), as well as the real version of them (see [20]).

On the chart $\mathrm{X}_0 \neq 0$ of $\mathbf{R}\mathrm{P}^n$, consider now the affine rescaled coordinates $\forall i \in \{1, \dots, n\}$, $x_i = \sqrt{d}\frac{\mathrm{X}_i}{\mathrm{X}_0}$. The associated renormalized and rescaled affine random polynomial $p$ defined by

$$\forall x \in \mathbf{R}^n, \quad p(x) := \frac{1}{\sqrt{d}^n} \mathrm{P}\left(1, \frac{x}{\sqrt{d}}\right)$$

satisfies

$$p(x) = \sum_{|\mathrm{J}| \leqslant d} a_{(d-|\mathrm{J}|, \mathrm{J})} \sqrt{\frac{(d+n)!}{(d-|\mathrm{J}|)! \mathrm{J}!}} \frac{x^{\mathrm{J}}}{\sqrt{d}^{n+|\mathrm{J}|}} \underset{d\to\infty}{\to} \sum_{\mathrm{J} \in \mathbf{N}^n} \tilde{a}_{\mathrm{J}} \frac{x^{\mathrm{J}}}{\sqrt{\mathrm{J}!}},$$

where the coefficients $(\tilde{a}_{\mathrm{J}})_{\mathrm{J} \in \mathbf{N}^n}$ are independent normal variables. In particular in dimension $n = 2$ the limit is precisely the random function $f$ given by (1.2). Moreover, the Fubini-Study $\mathrm{L}^2$-norm of the rescaled polynomials converges the norm (1.1). In fact, the more general process of random holomorphic sections of large powers of ample holomorphic line bundles over projective manifolds (and their real counterpart) rescale asymptotically to this universal process, too. This gives to the Bargmann-Fock space a universal algebraic origin.

*The Bogomolny-Schmidt conjecture.* — In [13] and [14], the authors studied the nodal lines of random sums of Gaussian waves in the real plane:

$$\forall x \in \mathbf{R}^2, \quad g(x) = \sum_{m=-\infty}^{\infty} a_m \mathrm{J}_{|m|}(r) e^{im\phi} \tag{1.7}$$

Here $(r, \phi)$ denotes the polar coordinates of $x$, $\mathrm{J}_k$ denotes the $k$-th Bessel function, and $(a_m)_{m\in\mathbf{Z}}$ are independent normal coefficients. The correlation function for this model equals (see [14])

$$e(x,y) = \mathbf{E}\big(g(x)g(y)\big) = \mathrm{J}_0(\|x-y\|), \tag{1.8}$$

hence is stationary, depends only on the distance between points, and decays polynomially in this distance, with degree $1/2$.

The authors associated to this random analytic model a discrete percolation-like model, and gave heuristic evidence that both model should be close. In particular, their work suggests that nodal lines should cross arbitrary large rectangles of fixed shape, with uniformly bounded probability. Numerical evidence was then obtained by several authors (see *e.g.* [12] and [8]).

As we will explain below, see Remark 4.8, our main result does not apply for this model since the decorrelation decay is too weak. Moreover, one of the main ingredient of the proof of Theorem 1.1 is the important FKG inequality given by condition 1. of Definition 2.1 below. Indeed in our case, it is the consequence of the positivity of the correlation function, but for random waves, the sign of the correlation function is not constant.

On the other hand, Theorem 1.6 applies to the model of random waves. Since the association of the model to a discrete percolation in [13] is based on two simplifications, namely the periodic distribution of the critical points and the independence of the process in space, Theorem 1.6 could be a more realistic mathematical way to handle the problem.

*Questions.* — From this present work natural questions arise.

- **What is the mean number of finite connected components per square?**
  In [34] and [35], the authors proved the existence of a non explicit asymptotic equivalence of the number of connected components of the nodal sets. In [25], [22], [21] and [24], the authors gave explicit lower and upper bounds for the expected Betti numbers of the nodal sets. However, these bounds are different, although the upper ones allow to get an asymptotic equivalent of the Euler characteristic (see [32]). In [13], the authors gave a heuristic way to compute the equivalence. Since their bond percolation has no correlation, they can use computations that give asymptotics for connected components of the percolation. Theorem 1.6 might possibly give new estimates of these constants, if discrete computations for correlated models can be performed.

- **Do Bargmann-Fock random nodal lines converge to $SLE(6)$?**
In a celebrated paper [43], Smirnov proved that for the triangular lattice, percolation interfaces converge, as the scale of the lattice tends to zero, to $SLE(6)$. Can we prove a similar result for Bargmann-Fock nodal lines? This is conjectured to hold, and supported by numerical evidence, in the case of random plane waves. In the case of percolation, RSW bounds are crucial, as is the obtention of a conformally invariant observable.
- **Do algebraic nodal lines behave like critical percolation?**
As explained before, our stationary model is in fact the universal model that arises in a very general algebraic situation. It is very likely that the following counterpart of Theorem 1.1 is true: *Fix a smooth bounded open set* U *in* $\mathbf{R}^2$ *and two arcs* $\gamma, \gamma'$ *in its boundary. Then, there exists a positive constant* $c$*, such that for any degree* $d$ *large enough, for a random polynomial*
$$p(x_1, x_2) = \sum_{i+j\leqslant d} a_{ij} \frac{x_1^i x_2^j}{\sqrt{(d-(i+j))!i!j!}}$$
*with i.i.d. Gaussian coefficients* $a_{ij}$*, with probability at least* $c$ *there is a connected component in* U *of the vanishing locus of* $p$ *touching both* $\gamma$ *and* $\gamma'$. This model converges to our analytic model, but unfortunately, it is no longer invariant by translation. This loss of symmetries, though small and vanishing in the limit, is fatal to some of the arguments we use.

*Structure of the article.* — In Sect. 2, we recall the main steps of Tassion's theorem in order to keep track of all the constants involved in the proofs. In Sect. 3, we obtain some simple results for general Gaussian fields and their link with the FKG inequalities. In Sect. 4, we first state a RSW theorem for the sign over a fixed lattice of a general correlated Gaussian with sufficiently correlation decay. Then we give the proof of Theorem 1.1, assuming that Theorem 1.6 is true. In the last Sect. 5, we prove Theorem 1.6, which shows that on large rectangles, the percolation process given by the sign of the random function is equivalent to the discrete percolation process given by the sign on the vertices of a lattice with sufficiently small mesh size, which is quantitatively estimated. In the appendix, we explain the general construction of the Wiener space associated to a Hilbert space, and we state a quantitative implicit function theorem that is used for the proof of Theorem 1.6.

## 2. Tassion's theorem

### 2.1. *Statement.*

*Notations, definitions and conditions.* — We consider a random process $\Omega$ on the plane $\mathbf{R}^2$, such that any point of $\mathbf{R}^2$ has a random color, black or white. In this article, we will consider two main processes. First, for any random Gaussian field $f$ on $\mathbf{R}^2$, $\Omega(f)$ will denote the coloring by sign (black if positive, white in the other case). Second, if $\mathcal{T}$ is a lattice, $\Omega(f, \mathcal{T})$ will denote the following coloring: a point $x$ in $\mathbf{R}^2$ is black if $x$ belongs to an edge between two vertices at which $f$ is positive, and white otherwise, see Definition 3.1.

- Recall that there exists a natural partial order on the elements of $\Omega$, choosing *white* < *black* at every point of $\mathbf{R}^2$. By definition, a *black-increasing* event $\mathcal{A}$ of the coloring process $\Omega$ is such that for any $\phi, \psi \in \Omega$ with $\phi \in \mathcal{A}$ and $\phi \leqslant \psi$, then $\psi \in \mathcal{A}$.
- For any $\rho \geqslant 1$ and $s > 0$, denote by $f_s(\Omega, \rho)$ be the probability that there exists a left-right black crossing in the rectangle $[0, \rho s] \times [0, s]$, that is a continuous curve of black points contained inside the rectangle and joining $\{0\} \times [0, s]$ to $\{\rho s\} \times [0, s]$.

*Definition* **2.1.** — *Fix a coloring process $\Omega$ as defined above, let us set the following conditions:*

1. *(FKG inequality) If $\mathcal{A}$, $\mathcal{B}$ are two black-increasing events, we have*
$$\mathbf{P}[\mathcal{A} \cap \mathcal{B}] \geqslant \mathbf{P}[\mathcal{A}]\mathbf{P}[\mathcal{B}].$$
2. *(Symmetries) The measure is invariant under $\mathbf{Z}^2$-translation, $\pi/2$-rotation centered at elements of $\mathbf{Z}^2$ and reflection with respect to the horizontal axis $(\mathrm{O}x)$.*
3. *(Percolation through squares) There exists $c_0(\Omega) > 0$ such that*
$$\forall s \in \mathbf{N}^*, f_s(\Omega, 1) \geqslant c_0(\Omega).$$

*When no confusion is possible, we will omit the reference to $\Omega$.*

We will use a recent theorem proved by V. Tassion, which establishes a RSW-type theorem in a general setting. Unfortunately, the statements in the article [44] cannot be directly applied here for two reasons.

- The first reason is not very fundamental, and can be omitted in a first reading. In [44], the two conditions 2. and 3. hold for any real translation and real sizes. More precisely, the analogous of condition 3. was the stronger condition

$$\forall s \geqslant 1, \quad f_s(\Omega, 1) \geqslant c_0(\Omega), \tag{2.1}$$

see [44, (1)]. Since we apply Tassion's theorem to the sign of a fixed function on lattices which are $\varepsilon$-rescaled copies of the Union Jack lattice $\mathcal{T}$, we can only get uniform percolation, in fact with probability $1/2$, on squares which are the union of fundamental squares of $\varepsilon\mathcal{T}$. Changing even a bit these squares changes a priori the probability of percolation, and it is not clear how. This is the reason why we replace (2.1) by 3.

- The second reason is crucial for our work. The article [44] is written for a fixed model, so that the constants arising in its theorems and their proofs are not in general explicit. However, as explained before, we need to apply these results for a family of processes $\Omega(f, \varepsilon\mathcal{T})$, see Definition 3.1 below, where the mesh will have a decreasing size $\varepsilon$. For this reason, we have to keep track all of the involved constants.

*Tassion's quantitative theorem.* — Here is a quantitative version of the main Theorem of [44]. We emphasize that this statement is essentially the one given by Tassion and does not add any fundamental information.

*Theorem* **2.2** *(see Tassion [44]). — For any* $\nu \in ]0, 1/2[$, *there exists a positive continuous function* $\mathrm{P}_\nu$ *defined on* $[1, +\infty[ \times ]0, 1[$, *such that for any model* $\Omega$ *satisfying the conditions of Definition 2.1, we have*

$$\forall \rho \geqslant 1,\ \forall s \in \mathbf{N}^*, \quad s \geqslant t_\nu(\Omega) \Rightarrow f_s(\Omega, \rho) \geqslant \mathrm{P}_\nu\big(\rho, c_0(\Omega)\big),$$

*where* $c_0(\Omega)$ *is given by condition 3. in Definition 2.1 and* $t_\nu(\Omega) \in [1, +\infty]$ *is given by formula (2.9) below.*

*Remark* **2.3.** — In [44], the result is stated without the restriction $s \geqslant t_\nu(\Omega)$, since, on one hand, $t_\nu(\Omega) < +\infty$ is implied by the additional condition (iii) of [44, Remark 2.2]. On the other hand, even if we know that $t_\nu(\Omega)$ is finite, we still need to uniformly estimate the probability of percolating in boxes of sizes smaller than $t_\nu(\Omega)$, which can be done easily for an individual process. However, these probabilities have no reason to be uniformly bounded from below over a collection of models.

The continuous model $\Omega(f)$ associated to the signs of the random analytic field with correlation $e_{\mathcal{W}}$ given by (1.4) satisfies the three conditions of Definition 2.1. However, there is no direct control of the parameter $t_\nu(\Omega)$, because this parameter is related to the independence of the field at different points. As explained before, in our case the analytic continuation principle means that local knowledge on the function is enough to determine it globally, so that a priori $t_\nu(\Omega)$ is infinite and Theorem 2.2 empty. To bypass this obstruction, we will discretize our continuous model, in order to extract the relevant information without discovering too much. In order to satisfy the symmetry assumption 2. in Definition 2.1, we choose a symmetric periodic triangulation invariant by horizontal reflection and $\pi/2$ rotation, for instance the face-centered square lattice, see Definition 3.3. We will consider a lattice with mesh size $\varepsilon > 0$. We will need to adapt this size to the size of the rectangles, see Theorem 1.6, which is the reason why we need to keep track of the constants in Tassion's theorem.

The rest of this section summarizes the principal lemmas of [44] and is devoted to the explanation of their quantitative refinements that we need for our own results. We do not repeat the whole proofs of the lemmas; instead we only explain in which way our weaker and quantitative conditions change them. The reader not interested in these technical refinements is advised to read Tassion's article [44] and then to go on with Sect. 3, where Theorem 4.7 gives an explicit bound for $t_\nu(\Omega)$ for the cases we are interested in. However, even if this bound is proved in Sect. 3, it depends on other parameters which we introduce in the rest of this present section. In particular, we introduce

- the function $\phi(\Omega, \cdot)$ given by (2.3),
- the function $\alpha(\Omega, \cdot)$ given by Lemma 2.7,
- the parameter $s(\Omega)$ given by (2.6),
- and the main parameter $t_\nu(\Omega)$ given by (2.9).

**2.2.** *Control of the constants.* — We consider in this subsection a random process $\Omega$ on the plane $\mathbf{R}^2$, such that any point of $\mathbf{R}^2$ has a random color, black or white, and which satisfies the three conditions of Definition 2.1. We follow the lemmas of [44], keeping track of the constants, in order to get the quantitative Theorem 2.2, and in particular an upper bound for the important parameter $t_\nu(\Omega)$. Moreover, we must take in account the weaker condition 3. in the proofs.

*Notation.* — We begin with further notations, which are chosen when possible to be close to the ones in [44].

- For any $n \in \mathbf{N}^*$ and any $s \geqslant 0$, define $B_s = [-s, s]^n \subset \mathbf{R}^n$.
- For $n = 2$ and any $0 \leqslant s \leqslant t < \infty$, $A_{s,t}$ denotes the annulus $B_t \setminus B_s$.
- For any $s > 0$, define $\mathcal{A}_s = \{$there exists a black circuit in the annulus $A_{s,2s}\}$.
- For any $S \subset \mathbf{R}^2$, $\sigma(S)$ denotes the sigma-algebra defined by the events measurable with respect to the coloring in S.
- For any pair (S, T) of subsets of $\mathbf{R}^2$, denote by $\phi(\Omega, S, T)$ the number

$$\phi(\Omega, S, T) = \sup_{\mathcal{A} \in \sigma(S), \mathcal{B} \in \sigma(T)} \left| \mathbf{P}[\mathcal{A} \cap \mathcal{B}] - \mathbf{P}[\mathcal{A}]\mathbf{P}[\mathcal{B}] \right|, \tag{2.2}$$

and by $\phi(\Omega, \cdot)$ the function

$$s \in \mathbf{R}_+^* \mapsto \phi(\Omega, s) = \phi\big(\Omega, A_{2s,4s}, B(s) \cup A_{5s, s\log s}\big), \tag{2.3}$$

see (iii) of [44, Remark 2.2].

*Remark* **2.4.**

- In particular, on an event with probability at least $1 - \phi(\Omega, s)$, the signs on $A_{2s,4s}$ and the signs on $B(s) \cup A_{5s, s\log s}$ can be coupled with the realization of a pair of independent colorings.

- The counterpart of the definition of $\phi(\Omega, s)$ in item (iii) of [44, Remark 2.2] is $\phi(\Omega, A_{2s,4s}, \mathbf{R}^2 \setminus A_{s,5s})$. For our percolation model, we could not obtain independence on subset of infinite area like $\mathbf{R}^2 \setminus A_{s,5s}$. On the other hand, Tassion's theorem needs only asymptotic independence between $A_{2s,4s}$ and $B(s) \cup A_{5s,C_1 s}$ for a certain fixed constant $C_1 > 0$, see Lemma 2.10 below, hence the presence of the $\log s$ term, which is larger than $C_1$ for $s$ large enough.
- Twice in our proofs, we will need $\phi(\Omega, \cdot)$ to have polynomial decay with sufficiently high degree, first in Tassion's argument in order to get sign percolation on large rectangles, see Lemma 2.10 below, and second in the topological argument given above: the existence of a percolating nodal line will be given by the percolation of a positive line and a parallel negative line. We must then know that the two latter events are almost independent, see the end of the proof of Proposition 4.11.
- In our case where the color is given by the sign of a random Gaussian field $f$, the polynomial decay of $\phi(\Omega, \cdot)$ is the consequence of the polynomial decay of the correlation function of $f$, see Proposition 4.1.

From now on to the end of this section, we explain how the proofs of [44] can be amended. We do not repeat the arguments, so that again, the interested reader may prefer to first read [44].

*Amending Tassion's lemmas.* — The following Lemma from [44] will be useful. It compares the probabilities of crossing a rectangle and the existence of a black circuit. We added the third assertion for a comparison of these probabilities between two different rectangles.

*Lemma* **2.5** *([44, Corollary 1.3]). — Let $s, s' \in \mathbf{N}^*$ with $s \leqslant s'$. Then*

1. $f_s(\Omega, 4)^4 \leqslant \mathbf{P}[\mathcal{A}_s] \leqslant f_s(\Omega, 2)$,
2. $f_s(\Omega, 1 + i\kappa) \geqslant f_s(\Omega, 1 + \kappa)^i f_s(\Omega, 1)^{i-1}$ *for any $\kappa > 0$ and any $i \geqslant 1$,*
3. *Let $\rho, \rho' > 1$. Then, $f_{s'}(\Omega, \rho') \geqslant f_s(\Omega, \rho)^{1+2\max(0, \frac{\rho' s' - \rho s}{\lfloor(\rho-1)s\rfloor})}$.*

*Proof.* — The last assertion is trivial if $\rho' s' \leqslant \rho s$, since then $f_{s'}(\Omega, \rho') \geqslant f_s(\Omega, \rho)$. In the other case, using the condition 2. of Definition 2.1, it can be proved using a sequence of rectangles translated and $\pi/2$-rotated from $[0, \rho s] \times [0, s]$, alternatively horizontal and vertical, in order to cross the larger rectangle $[0, \rho' s'] \times [0, s']$. □

*Remark* **2.6.** — If the lower bounds $f_s(\Omega, \rho)$ given by Theorem 2.2 for an integer $s$ hold for rectangles translated by elements of $\mathbf{R}^2$, and not only by elements of $\mathbf{Z}^2$, then by similar arguments, we can replace $s \in \mathbf{N}^*$ by $s \geqslant 1$ in the statement of Lemma 2.5.

The quantitative parameter $t_\nu(\Omega)$ of Theorem 2.2 is itself related to the important function $s \mapsto \alpha(\Omega, s)$ defined in [44, Lemma 2.1]. We recall its definition. For this, let $s \geqslant 1$ and $-s/2 \leqslant \alpha \leqslant \beta \leqslant s/2$, and let us set (see Fig. 2 for an illustration)

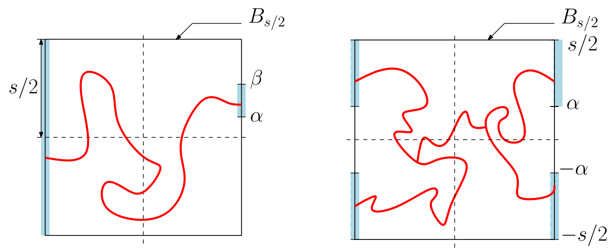


Fig. 2. — The events $\mathcal{H}_s(\alpha,\beta)$ and $\mathcal{X}_s(\alpha)$ (from [44])

- $\mathcal{H}_s(\alpha,\beta)$ to be the event that there exists a black path in the square $B_{s/2}$, from the left side to $\{s/2\}\times[\alpha,\beta]$;
- $\mathcal{X}_s(\alpha)$ to be the event that there exists in $B_{s/2}$ a black path $\gamma_{-1}$ from $\{-s/2\}\times[-s/2,-\alpha]$ to $\{-s/2\}\times[\alpha,s/2]$, a black path $\gamma_1$ from $\{s/2\}\times[-s/2,-\alpha]$ to $\{s/2\}\times[\alpha,s/2]$, and a black path from $\gamma_{-1}$ to $\gamma_1$.

*Lemma* **2.7** *([44, Lemma 2.1]). — There exists a universal polynomial* $Q_1\in\mathbf{R}[X]$, *positive on* $]0,1[$, *such that for every* $s\in 2\mathbf{N}^*$, *there exists* $\alpha(\Omega,s)\in[0,s/4]$ *satisfying the following two properties:*

**(P1)** $\mathbf{P}[\mathcal{X}_s(\alpha(\Omega,s))]\geqslant Q_1(c_0(\Omega))$.

**(P2)** *If* $\alpha(\Omega,s)<s/4$, *then* $\mathbf{P}[\mathcal{H}_s(0,\alpha(\Omega,s))]\geqslant c_0(\Omega)/4+\mathbf{P}[\mathcal{H}_s(\alpha(\Omega,s),s/2)]$.

*Proof*. — In [44], $\alpha(\Omega,s)$ is denoted by $\alpha_s$, and $Q_1(c_0)$ by $c_1$, which is equal to $c_0(c_0/8)^4$. The main difference between our setting and Tassion's is that our a priori bounds only hold for squares of integer sizes, which is why we state the lemma only for $s$ in $2\mathbf{N}^*$. The argument is otherwise exactly the same. □

The following lemma allows to get a black circuit (and hence a percolation in a rectangle by Lemma 2.5) at any large scale, if we have a good control of the function $\alpha(\Omega,\cdot)$. First, for any $s\in\mathbf{N}^*$, define

$$k(s)\in\{0,\dots,5\},\qquad s+k(s)\equiv 0 \bmod [6]. \tag{2.4}$$

*Lemma* **2.8** *([44, Lemma 2.2]). — There exist two universal polynomial* $q_2,\tilde q_2\in\mathbf{R}[X]$, *positive on* $]0,1[$, *such that if* $Q_2(c_0):=\min(q_2(c_0),\tilde q_2(c_0))$, *then*

$$\forall s\in 3\mathbf{N}^*,\quad \alpha\big(\Omega,s+k(s)\big)\leqslant 2\big\lfloor\alpha(\Omega,2s/3)\big\rfloor\Rightarrow \mathbf{P}[\mathcal{A}_{s+k(s)}]\geqslant Q_2(c_0),$$

*Proof*. — First, we prove the lemma without the floors, the integer condition on $s$ and with $k(s)=0$. In [44], $Q_2(c_0)$ is denoted by $c_2$. The proof of [44, Lemma 2.2] gives that either

$$\mathbf{P}[\mathcal{A}_s]\geqslant\big(Q_1(c_0)^{15}c_0^2\big)^4:=q_2(c_0),$$

or $f_s(\Omega, 4/3) \geqslant Q_1(c_0)(c_0/4)^2$. By Lemma 2.5, in this case

$$\mathbf{P}[\mathcal{A}_s] \geqslant \left(f_s(\Omega, 4/3)^9 c_0^8\right)^4 \geqslant \left(\left(Q_1(c_0)(c_0/4)^2\right)^9 c_0^8\right)^4 := \tilde{q}_2(c_0).$$

Now, we add the floors, the integer condition and the integer $k(s)$. In the proof of [44, Lemma 2.2], we change $\alpha_{2s/3}$ into $\lfloor \alpha_{2s/3} \rfloor$ any time it appears, and we change a bit the square R into

$$R = \left(-\frac{1}{6}\big(s + k(s)\big), -\lfloor \alpha_{2s/3} \rfloor\right) + B_{(s+k(s))/2},$$

while $R' = ((s + k(s))/6, -\lfloor \alpha_{2s/3} \rfloor) + R_{(s+k(s))/2}$. Then thanks to these floors and the restriction $s + k(s) \in 6\mathbf{N}$, R, R′ and $B_{s/3}$ have vertices with integral coordinates, so that condition 3. of Definition 2.1 can be applied. Moreover, since

$$\alpha\big(\Omega, s + k(s)\big) \leqslant 2\big\lfloor \alpha(\Omega, 2s/3) \big\rfloor \leqslant 2\alpha(\Omega, 2s/3),$$

we still get the intersection of the events $\mathcal{X}(\alpha(\Omega, 2s/3))$ in $B_{s/3}$ and with the event $\mathcal{E}$ and $\mathcal{E}'$, with the same probabilities without the integer and floor additions. □

*Lemma* **2.9** *([44, Lemma 3.1]). — There exists a universal continuous positive function* $Q_3$ *defined on* $]0, 1[$, *such that the following holds: for every* $s \geqslant 1$ *and* $t \in 2\mathbf{N}^*$, $t \geqslant 4s$, *if* $\mathbf{P}[\mathcal{A}_s] \geqslant Q_2(c_0)$ *and* $\alpha(\Omega, t) \leqslant s$, *then* $\mathbf{P}[\mathcal{A}_t] \geqslant Q_3(c_0)$.

*Proof*. — In [44], $Q_3(c_0)$ is denoted by $c_3$, and can be chosen to be

$$Q_3(c_0) := \left(Q_2(c_0)(c_0/4)^2\right)^3 c_0^2,$$

with $Q_2$ given by Lemma 2.8. The integer constraint allows to apply the proof since the restricted condition 3. then applies for the boxes $[-t, 0] \times [-t/2, t/2]$ and $[0, t] \times [-t/2, t/2]$. □

Now, define

$$\begin{aligned} \tau_1 : \,]0, 1[ &\to [4, \infty[ \\ c_0 \mapsto \tau_1(c_0) &= \max\left\{4, \exp\left[\frac{\log 5 \log(c_0/8)}{\log(1 - Q_3(c_0)/2)} + \log 5\right]\right\}, \end{aligned} \tag{2.5}$$

where $Q_3$ is defined by Lemma 2.9. In [44, Lemma 3.2], $\tau_1(c_0)$ is denoted by $C_1$, where $C_1$ is any constant satisfying $C_1 \geqslant 4$ and

$$(1 - c_3/2)^{\lfloor \log_5(C_1) \rfloor} < c_0/4,$$

with $c_3 = Q_3(c_0)$, see (5) in [44, Lemma 3.2]. Note that we replaced the right-hand side $c_0/4$ by $c_0/8$ for our definition of $\tau_1$. This is due to the proof of our version of Lemma 2.10 of [44, Lemma 3.2], see below. Now, define the integer

$$s(\Omega) = \max\Big\{ s \in \mathbf{N}^*,\ s \geqslant \exp\big(\tau_1(c_0)\big),\ \phi(\Omega, s) \geqslant \frac{c_0}{16} Q_3(c_0) \Big\}, \tag{2.6}$$

where $\phi(\Omega, \cdot)$ is defined by (2.3) and $Q_3$ is given by Lemma 2.9. Note that

$$\forall s \geqslant s(\Omega), \quad \sup_{\substack{\mathcal{A} \in \sigma(A_{2s,4s}) \\ \mathcal{B} \in \sigma(B(s) \cup A_{5s, \tau_1(c_0)s})}} |\mathbf{P}[\mathcal{A} \cap \mathcal{B}] - \mathbf{P}[\mathcal{A}]\mathbf{P}[\mathcal{B}]| \leqslant \frac{c_0}{16} Q_3(c_0). \tag{2.7}$$

The parameter $s(\Omega)$ estimates the scale $s$ from which events on rings of size of order $s$ and separated from each other by $s$ are almost independent, see Remark 2.4. In [44], $s(\Omega)$ is denoted by $s_0$, without the condition of being an integer and larger than $\exp(\tau_1)$, and related to the particular event $\mathcal{F}_s$ associated to Voronoi percolation, see condition (4) before [44, Lemma 3.2].

*Lemma* **2.10** *([44, Lemma 3.2]). — For any $s \in \mathbf{N}^*$, $s \geqslant s(\Omega)$, if $\mathbf{P}[\mathcal{A}_s \geqslant Q_2(c_0)]$, then there exists $s' \in [4s, \tau_1(c_0)s] \cap \mathbf{N}^*$ such that $\alpha(\Omega, s') \geqslant s$.*

*Proof.* — The proof is almost the same, but we replace the inequality (7) in [44] by $\mathbf{P}[\mathcal{A}_{5^i s}] \geqslant Q_3(c_0)$ and (8) by

$$\mathbf{P}\big[\mathcal{E}^c\big] \leqslant \mathbf{P}\Big[\bigcap_{1 \leqslant i \leqslant \lfloor \log_5(C_1) \rfloor} \mathcal{A}^c_{5^i s}\Big].$$

We apply inequality (2.7) in order to get

$$\begin{aligned}
&\mathbf{P}\Big[\bigcap_{1 \leqslant i \leqslant \lfloor \log_5(\tau_1) \rfloor} \mathcal{A}^c_{5^i s}\Big] \\
&\quad \leqslant \mathbf{P}[\mathcal{A}^c_{5s}]\mathbf{P}\Big[\bigcap_{2 \leqslant i \leqslant \lfloor \log_5(\tau_1) \rfloor} (\mathcal{A}_{5^i s})^c\Big] + \frac{c_0}{16} Q_3 \leqslant \cdots \\
&\quad \leqslant \prod_{1 \leqslant i \leqslant \lfloor \log_5(\tau_1) \rfloor} \mathbf{P}[\mathcal{A}^c_{5^i s}] + \frac{c_0}{16} Q_3 \sum_{0 \leqslant i \leqslant \lfloor \log_5(\tau_1) \rfloor - 2} \prod_{1 \leqslant j \leqslant i} \mathbf{P}[\mathcal{A}^c_{5^j s}] \\
&\quad \leqslant (1 - Q_3/2)^{\lfloor \log_5(\tau_1) \rfloor} + \frac{c_0}{16} Q_3 \sum_{0 \leqslant i \leqslant \lfloor \log_5(\tau_1) \rfloor - 2} (1 - Q_3/2)^i \\
&\quad < c_0/4.
\end{aligned}$$

We used (2.5) in the last inequality. The formulation with the integer condition is straightforward since in the proof $5^i s \in \mathbf{N}^*$, and we just choose $\lfloor C_1 s \rfloor$ instead of $C_1 s$. $\square$

We can now define the main quantitative parameter $t_\nu(\Omega)$. For any $\nu \in ]0, 1/2[$, set

$$\gamma(\nu) = 1 + \log_{4/(3+2\nu)}(3/2 + \nu) > 1 \tag{2.8}$$

and

$$t_\nu(\Omega) = (3/2 + \nu) s_\nu(\Omega)^{\gamma(\nu)} \alpha\big(\Omega, s_\nu(\Omega)\big)^{1-\gamma(\nu)}, \tag{2.9}$$

where

$$s_\nu(\Omega) = \max\big(s(\Omega), \lfloor 6/\nu \rfloor + 1\big), \tag{2.10}$$

see (2.6) for the definition of $s(\Omega)$ and Lemma 2.7 for $\alpha(\Omega, \cdot)$. In the rest of the article, we will write $\alpha(\cdot)$ instead of $\alpha(\Omega, \cdot)$ when the process $\Omega$ is explicit.

*Lemma* **2.11** *([44, Lemma 3.3]). — For any $\tau \in ]0, 1/2[$, there exists a continuous universal function $\tau_{3,\nu} : ]0, 1[ \to [4, \infty[$ and an infinite sequence $(s_i)_{i \in \mathbf{N}^*} \in (6\mathbf{N}^*)^{\mathbf{N}^*}$ such that*

- $s_1 \leqslant t_\nu(\Omega)$,
- $\forall i \geqslant 1,\ 4s_i \leqslant s_{i+1} \leqslant \tau_{3,\nu}(c_0) s_i$,
- *and* $\mathrm{P}[\mathcal{A}_{s_i}] > \mathrm{Q}_2(c_0)$.

*Proof*. — In order to obtain the existence and an estimate of $s_1$, let us define the following sequence $(\sigma_p)_{p \in \mathbf{N}}$: $\sigma_0 = s_\nu(\Omega) \in \mathbf{N}^*$, see (2.10), and for any $p \in \mathbf{N}^*$,

$$\sigma_{p+1} = \frac{3}{2}\sigma_p + k\left(\frac{3}{2}\sigma_p\right) \in 6\mathbf{N}^*$$

where the function $k$ is defined by (2.4). For any $s \geqslant 6/\nu$, $k(s) \leqslant \nu s$, so that

$$\forall p \in \mathbf{N}, \quad \sigma_p \leqslant (3/2 + \nu)^p s_\nu(\Omega).$$

Denote by N the first $p \in \mathbf{N}^*$, such that $\alpha(\Omega, \sigma_{p+1}) \leqslant 2\alpha(\Omega, \sigma_p)$. Then

$$(3/2 + \nu)^{\mathrm{N}} s_\nu(\Omega) \geqslant \sigma_{\mathrm{N}} > \alpha(\Omega, \sigma_{\mathrm{N}}) > 2^{\mathrm{N}} \alpha\big(s_\nu(\Omega)\big)$$

so that

$$\mathrm{N} \leqslant \log\left(\frac{s_\nu(\Omega)}{\alpha(s_\nu(\Omega))}\right) \log^{-1}\left(\frac{4}{3 + 2\nu}\right).$$

Choose

$$s_1 := \sigma_{\mathrm{N}+1} \leqslant (3/2 + \nu)^{\mathrm{N}+1} s_\nu(\Omega).$$

Then $s_1 \leqslant t_\nu(\Omega)$, by (2.9). Moreover by Lemma 2.8, we have $\mathbf{P}[\mathcal{A}_{s_1}] \geqslant \mathrm{Q}_2(c_0)$. The existence of the rest of the sequence $(s_i)_{i \geqslant 2}$ follows the same lines than [44]. If $\nu = 0$, $\tau_{3,0}(c_0)$

is denoted by $C_3$ by Tassion and is defined by $C_3 = C_1^{1+\log_{4/3}(3/2)}$, where $C_1$ is our $\tau_1(c_0)$ defined by Lemma 2.10. In the case $\nu > 0$ we just change $C_3$ into

$$\tau_{3,\nu}(c_0) = C_1^{1+\log_{4/(3+2\nu)}(3/2+\nu)}.$$

This concludes the proof. $\square$

*Proof of Theorem 2.2.* — If $(\rho - 1)s < 1$, then $f_s(\Omega, \rho) = c_0$ and there is nothing to prove. From now on, we assume that the converse holds, which implies that $\lfloor(\rho - 1)s\rfloor \geqslant (\rho - 1)s/2$. For any $i \geqslant 1$, we want to prove uniform percolation for integer sizes between $s_i$ and $s_{i+1}$. By assertions 3. and 1. of Lemma 2.5, for all integer $i \geqslant 1$,

$$f_{s_i}(\Omega, \rho) \geqslant f_{s_i}(\Omega, 2)^{1+2\max(0,\rho-2)} \geqslant \mathbf{P}[\mathcal{A}_{s_i}]^{1+2\max(0,\rho-2)} \geqslant Q_2(c_0)^{1+2\max(0,\rho-2)},$$

so that by the same assertion, for any $s \in [s_i, s_{i+1}] \cap \mathbf{N}^*$,

$$f_s(\Omega, \rho) \geqslant f_{s_i}(\Omega, \rho)^{1+2\max(0, \frac{\rho(s_{i+1}-s_i)}{\lfloor(\rho-1)s_i\rfloor})} \geqslant Q_2(c_0)^{(1+4\max(0,\rho-2))(1+2(\tau_{3,\nu}(c_0)-1)\frac{\rho}{\rho-1})}.$$

The right-hand side can be chosen to be the universal function $P_\nu(\rho, c_0)$. $\square$

## 3. Gaussian fields

In this section we introduce some natural conditions for the Gaussian fields we will work with, and we prove some more or less elementary percolation properties on the associated processes on lattices or on $\mathbf{R}^2$.

**3.1.** *Various conditions.* — Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a centered Gaussian field, that is for every $x \in \mathbf{R}^n$, $f(x)$ is a random centered Gaussian variable. Let $\mathcal{T}$ be a lattice in $\mathbf{R}^2$, $\mathcal{E}$ its set of edges and $\mathcal{V}$ its set of vertices.

*Definition* **3.1.** — *Let us define the two following processes on* $\mathbf{R}^2$*:*

- $\Omega(f)$ *denotes the coloring by sign, that is* $x \in \mathbf{R}^2$ *is black iff* $f(x) > 0$ *and white otherwise.*
- $\Omega(f, \mathcal{T})$ *denotes the following coloring on* $\mathbf{R}^2$*: if* $x \in \mathbf{R}^2$ *is a vertex where* $f$ *is positive, or if it belongs to an edge* $e \in \mathcal{E}$ *whose two extremities* $v_1 \in \mathcal{V}$ *and* $v_2 \in \mathcal{V}$ *satisfy* $f(v_1) > 0$ *and* $f(v_2) > 0$, *then* $x$ *is black; otherwise* $x$ *is white.*

*Remark* **3.2.** — Since outside a set of vanishing measure, $f$ does not vanish on $\mathcal{V}$, a.s. a vertex $v \in \mathcal{V}$ is white if and only if $f(v) < 0$.

In the proof of the main Theorem 1.1 (but not Theorem 1.6), we will need lattices with the following properties.

*Definition* **3.3.** — *For any lattice $\mathcal{T} \subset \mathbf{R}^2$, define the following conditions:*

- *(Periodicity) $\mathcal{T}$ is periodic,*
- *(Triangulation) $\mathcal{T}$ is a triangulation,*
- *(Symmetry) $\mathcal{T}$ is invariant by the reflection with respect to the horizontal axis, and by $\pi/2$-rotation around one of its vertices.*
- *(Integrality) There exists $\mathrm{N} \in \mathbf{N}^*$, such that $\mathcal{V} \subset (\frac{1}{\mathrm{N}}\mathbf{Z})^2$.*

*Remark* **3.4.**

- As a lattice satisfying the symmetry conditions, one can choose the face-centered square lattice, though the specific choice will not be relevant in our proofs.
- The triangulation condition is needed for duality reasons, only to ensure condition 3. of Definition 2.1 for $\Omega(f, \mathcal{T})$, see Lemma 4.4.

Recall that the correlation function $e$ of $f$ is defined by $e(x,y) = \mathbf{E}(f(x)f(y))$ for any $(x,y)$ in $(\mathbf{R}^n)^2$. Depending on the nature of the results, we will require the field $f$ to fulfill part or all of the following conditions.

- *(Stationarity)* There exists a function $\mathrm{K} : \mathbf{R}^n \to \mathbf{R}$ such that

$$\forall x, y \in \mathbf{R}^n, \quad e(x,y) = \mathrm{K}(x-y). \tag{3.1}$$

  In this case, we normalize $f$ so that $\forall x \in \mathbf{R}^n$, $\mathrm{K}(0) = e(x,x) = 1$.
- *(Positivity)*

$$\mathrm{K} \geqslant 0. \tag{3.2}$$

- *(Symmetry when $n = 2$)*

$$\forall (a,b) \in \mathbf{R}^2, \quad \mathrm{K}\big((a,-b)\big) = \mathrm{K}\big((a,b)\big) = \mathrm{K}\big((b,a)\big). \tag{3.3}$$

- *(Polynomial decay)*

$$\exists \alpha > 0,\ \exists \beta,\ \forall x \in \mathbf{R}^n, \quad |\mathrm{K}(x)| \leqslant \beta \|x\|^{-\alpha}. \tag{3.4}$$

- *(Non-degeneracy)* The quadratic form

$$v \in \mathbf{R}^n \mapsto d^2\mathrm{K}(0)(v,v) \tag{3.5}$$

  is negative definite.

Condition 3.5 is needed for the quantitative transversality of the field, in particular in Lemma 5.4 and Lemma 5.5.

*Remark* **3.5.** — For a stationary Gaussian field with a $C^2$ covariance, condition (3.4) for positive $\alpha$ implies condition (3.5). Indeed, assume that for some $v \in \mathbf{R}^n \setminus \{0\}$ one has $d^2\mathrm{K}(0)(v, v) = 0$. Then, as $\varepsilon$ tends to zero,

$$\mathbf{E}\Big[\big(f(\varepsilon v) - f(0)\big)^2\Big] = 2\big(1 - \mathrm{K}(\varepsilon v)\big) = o\big(\varepsilon^2\big).$$

This implies that $\mathbf{E}[|f(\varepsilon v) - f(0)|] = o(\varepsilon)$ and by the triangle inequality,

$$\mathbf{E}\big[|f(v) - f(0)|\big] \leqslant \sum_{k=0}^{n-1} \mathbf{E}\left[\left|f\left(\frac{(k+1)v}{n}\right) - f\left(\frac{kv}{n}\right)\right|\right] = o(1),$$

which contradicts condition (3.4) in the direction $v$, unless $f$ is identically zero.

*Lemma* **3.6.** — *For any $\alpha > 0$ and any $m \in \mathbf{N}$, the stationary centered Gaussian field associated to the real Bargmann-Fock space, with correlation $e_{\mathcal{W}}(x,y) = \exp(-\frac{1}{2}\|x-y\|^2)$, see (1.4), satisfies all of these hypotheses.*

*Proof.* — Only the last condition has to be clarified. If $\mathrm{Z} \in \mathbf{R}^n$ is a normal Gaussian vector, then

$$\forall x \in \mathbf{R}^n, \quad \mathrm{K}(x) = \exp\left(-\frac{1}{2}\|x\|^2\right) = \mathbf{E}\big[e^{i\langle x, \mathrm{Z}\rangle}\big],$$

so that for any $v \in \mathbf{R}^n$,

$$d^2\mathrm{K}(0)(v, v) = -\mathbf{E}\big[\langle v, \mathrm{Z}\rangle^2\big]$$

which is negative if $v \neq 0$. Hence, condition (3.5) holds. □

*Remark* **3.7.** — Note that if $f$ is stationary ($f$ satisfies (3.1)), the covariance function $e$ is symmetric, that is $\forall (x,y) \in (\mathbf{R}^2)^2$, $e(x,y) = e(y,x) = \mathrm{K}(x-y)$, so that any odd derivative of K at 0 vanishes. In particular, two derivatives of different parity of $f$ are independent random variables.

**3.2.** *The FKG inequality.* — One of the crucial ingredients of the proof of Theorem 1.1, is the Fortuin-Kasteleyn-Ginibre (FKG) inequality for our model, see Section 2.1 and condition 1. in Definition 2.1. The following theorem gives a link between positive correlation and FKG inequality.

*Theorem* **3.8** *(see Pitt [37]).* — *Let $\varphi = (\varphi_i)_{1\leqslant i\leqslant k}$ be a Gaussian vector in $\mathbf{R}^k$ with nonnegative covariance function. Then $\varphi$ satisfies the FKG property: for any two bounded, nondecreasing, measurable functions* $\mathrm{F}, \mathrm{G} : \mathbf{R}^k \to \mathbf{R}$, *one has*

$$\mathrm{E}\big[\mathrm{F}(\varphi)\mathrm{G}(\varphi)\big] \geqslant \mathrm{E}\big[\mathrm{F}[\varphi]\big]\mathrm{E}\big[\mathrm{G}[\varphi]\big]. \tag{3.6}$$

*In particular, if $\mathcal{A}$ and $\mathcal{B}$ are increasing events on $\varphi$, then* $\mathrm{P}[\mathcal{A} \cap \mathcal{B}] \geqslant \mathrm{P}[\mathcal{A}]\mathrm{P}[\mathcal{B}]$.

In our case, $\phi$ will denote the sign of $f$ at the vertices of a lattice, and typically $\mathcal{A}$ and $\mathcal{B}$ will denote a black crossing of a rectangle, or the existence of a black circuit in a ring.

*Remark* **3.9.** — This theorem will be enough for our purposes here, because all the events we will consider will actually depend on the value of the random field $f$ at finitely many points; but the results readily extend by approximation to continuous increasing functionals of the whole field.

The following lemma is an immediate corollary of this theorem.

*Lemma* **3.10.** — *For any stationary random Gaussian field $f$ satisfying condition (3.2) (positivity) and any lattice $\mathcal{T}$, the processes $\Omega(f)$ and $\Omega(f, \mathcal{T})$ introduced in Definition 3.1 satisfy condition 1. (FKG) of Definition 2.1.*

*Remark* **3.11.** — In the particular case of the Gaussian field given by the real analytic functions in the Wiener space $\mathcal{W}$, see (1.4) or (1.2), the expansion of the field in a monomial basis gives a decomposition of the form $f = \sum_{i\in \mathrm{I}} a_i f_i$ where the $a_i$ are i.i.d. Gaussian variables, and where the $f_i$ are positive in the first quadrant. A monotone functional of $f$ depending only on the values of $f$ in that quadrant can then be seen as a monotone functional of the $(a_i)_{i\in \mathrm{I}}$, and the inequality (3.6) follows directly from the statement of the Harris-FKG inequality for a product measure. Going from the quadrant to the whole space then follows from stationarity.

**3.3.** *Elementary percolation.* — We will need the following two simple lemmas in the proof of the main results. The first one is a direct consequence of continuity and monotone convergence:

*Lemma* **3.12** *(Uniform percolation on a small box).* — *Let $f$ be a $\mathrm{C}^0$ stationary Gaussian field on $\mathbf{R}^n$. Then if $\mathrm{B}_\lambda = [-\lambda, \lambda]^n$,*

$$\mathbf{P}[f_{|\mathrm{B}_\lambda} > 0] \underset{\lambda\to 0}{\to} \mathbf{P}\big[f(0) > 0\big] = 1/2.$$

*Remark* **3.13.** — If moreover $f$ is assumed to be $\mathrm{C}^3$, one can give a quantitative estimate of the convergence rate. Fix $u > 0$, such that $f(0) \geqslant u$ with probability at least $1/2 - \delta/2$. By Markov inequality and Lemma 5.2 below applied to $p = 1$,

$$\forall s \leqslant 2, \quad \mathbf{P}\left[\|f\|_{\mathrm{C}^1(\mathrm{B}_\lambda)} \leqslant \frac{u}{2\sqrt{2}\lambda}\right] \geqslant 1 - \left(\frac{2\sqrt{2}\lambda}{u}\right)(\mathrm{C}_1\sqrt{\log 2}).$$

If $\lambda = (\frac{\delta}{2})u(2\sqrt{2}\mathrm{C}_1\sqrt{\log 2})^{-1}$, the two events happen simultaneously with probability larger than $1/2 - \delta$, and in this case by Taylor applied between 0 and any point of $\mathrm{B}_\lambda$, we obtain $f_{|\mathrm{B}_\lambda} > 0$.

The following Lemma asserts that trivially the percolation process associated to $f$ has uniform positive probability of happening in rectangles inside a fixed square.

*Lemma* **3.14** *(Uniform percolation on a box). — Let $f$ be a $C^0$ stationary Gaussian field on $\mathbf{R}^2$ satisfying condition (3.2) (positivity). Then*

$$\forall s > 0, \quad \mathbf{P}[f_{|B_s} > 0] > 0.$$

*Proof.* — By stationarity and by Lemma 3.12 there exists $\lambda > 0$ such that for any fixed box $b$ which is a translation of $B_\lambda$, with probability at least $1/4$ we have $f_{|b} > 0$. Since being positive on a box is a increasing event, the FKG property given by Theorem 3.8 implies that the wanted probability is larger than $1/4^N$, where N denotes the number of cubes like $b$ needed to cover $B_s$. □

*Remark* **3.15.** — Note that if $f$ satisfies the conditions of Lemma 3.14, then

$$\forall s > 0,\ \exists a > 0,\ \forall \mathcal{T}, \quad \mathbf{P}[f_{|\mathcal{V}\cap B_s} > 0] \geqslant a,$$

where $\mathcal{T} \subset \mathbf{R}^2$ is a lattice and $\mathcal{V}$ its set of vertices.

## 4. Discretized percolation

We begin with a discrete correlated percolation problem on a periodic lattice $\mathcal{T}$. Then we explain how to transfer our continuous percolation problem to a discrete one.

**4.1.** *A correlated discrete percolation.* — Let $\mathcal{T}$ be a periodic graph, $\mathcal{V}$ be its set of vertices, $\mathcal{E}$ be its set of edges and $a_\mathcal{T}$ be the asymptotic density of vertices of $\mathcal{V}$, *i.e.* the mean number of vertices in a randomly translated unit square.

*Quantitative independence.* — Let $f$ be a centered Gaussian field on $\mathcal{V}$. We give a quantitative estimate for the function $\phi(\Omega(f, \mathcal{T}), \cdot)$ defined by (2.3). Recall that this function estimates the independence of events in two disjoints rings and is obtained by a union bound. Note that this parameter can be defined without any assumptions on $f$ or $\mathcal{T}$.

*Proposition* **4.1.** — *There exists a constant* $C > 0$*, such that for any random Gaussian stationary field $f$ on any lattice $\mathcal{T}$, for any pair of bounded measurable subsets* $(S, T)$ *of* $\mathbf{R}^2$,

$$\phi\big(\Omega(f, \mathcal{T}), S, T\big) \leqslant C a_\tau^{8/5} \mathit{Area}(S \cup T)^{8/5} \sup_{(v,w)\in S\times T} |K(v - w)|^{1/5}, \tag{4.1}$$

*where* K *is defined by (3.1).*

The following corollary is a straightforward consequence of Proposition 4.1.

*Corollary* **4.2.** — *If $f$ satisfies in addition the condition (3.4) (polynomial decay with degree $\alpha > 0$), then there exists $C' > 0$ depending only on $C$ and the constants $\alpha$, $\beta$ of condition (3.4) such that for any lattice $\mathcal{T}$ and any pair of bounded measurable subsets $(S, T)$ of $\mathbf{R}^2$,*

$$\phi\big(\Omega(f,\mathcal{T}), S, T\big) \leqslant C' a_{\mathcal{T}}^{8/5} \mathrm{area}^{8/5}(S \cup T)\big(\mathrm{dist}(S,T)\big)^{-\frac{\alpha}{5}} \quad \textit{and}$$
$$\forall s \geqslant 1, \quad \phi\big(\Omega(f,\mathcal{T}), s\big) \leqslant C' a_{\mathcal{T}}^{8/5} s^{\frac{16-\alpha}{5}} \log^{16/5} s.$$

In order to prove Proposition 4.1, we begin with the following Theorem 4.3 below, which quantifies the dependence between the two components of an orthogonal decomposition of a Gaussian vector, to be the two vectors made of the values of the Gaussian field on the vertices of $S \cap \mathcal{T}$ and $T \cap \mathcal{T}$. This theorem has its own interest.

*Theorem* **4.3.** — *There is a universal positive constant $C$ such that the following holds. Let $X$ and $Y$ be two Gaussian vectors in $\mathbf{R}^{m+n}$, respectively of covariance*

$$\Sigma_X = \begin{bmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_2 \end{bmatrix} \quad \textit{and} \quad \Sigma_Y = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

*where $\Sigma_1 \in M_m(\mathbf{R})$ and $\Sigma_2 \in M_n(\mathbf{R})$ have all diagonal entries equal to $1$. Denote by $\mu_X$ (resp. $\mu_Y$) the law of the signs of the coordinates of $X$ (resp. $Y$), and by $\eta$ the largest absolute value of the entries of $\Sigma_{12}$. Then,*

$$d_{TV}(\mu_X, \mu_Y) \leqslant C(m+n)^{8/5}\eta^{1/5}.$$

*In particular, if $A$ (resp. $B$) is an event depending only on signs of the first $m$ (resp. on the last $n$) coordinates of $X$, then*

$$|P[A \cap B] - P[A]P[B]| \leqslant C(m+n)^{8/5}\eta^{1/5}.$$

*Proof.* — Let $\lambda$ and $\varepsilon$ be positive constants, to be chosen later. Write $X = (X_1, X_2)$ where $X_1 \in \mathbf{R}^m$ and $X_2 \in \mathbf{R}^n$; we focus for the moment on $X_1$. Each of its coordinates is a normal random variable, and therefore has absolute value larger than $\varepsilon$ with probability at least $1 - \varepsilon$. Therefore, by a union bound, outside of an event $E_1$ of probability at most $m\varepsilon$, all the entries of $X_1$ have absolute value at least $\varepsilon$.

Let $(x_1, \dots, x_m)$ be an orthonormal basis of $\mathbf{R}^m$ diagonalizing $\Sigma_1$, and ordered in such a way that the eigenvalues $\lambda_k$ corresponding to $x_k$ for $k \leqslant m_0$ (resp. $k > m_0$) are at least equal to $\lambda$ (resp. smaller than $\lambda$) in absolute value; one can write

$$X_1 = \sum_{k=1}^{m} a_k \lambda_k^{1/2} x_k$$

where the $a_k$ are independent normal variables. By the Markov inequality, for $k > m_0$:

$$\mathbf{P}\big[\big\| a_k \lambda_k^{1/2} x_k \big\|_2 > \varepsilon/2m\big] \leqslant \frac{\mathbf{E}[\| a_k \lambda_k^{1/2} x_k \|_2^2]}{(\varepsilon/2m)^2} \leqslant \frac{4\lambda m^2}{\varepsilon^2}.$$

Therefore, outside an event $F_1$ of probability at most $4\lambda m^3/\varepsilon^2$, the bound $\|a_k\lambda_k^{1/2}x_k\|_2 \leqslant \varepsilon/2m$ holds for all $k > m_0$ and therefore the vector

$$\tilde{X}_1 := \sum_{k=1}^{m_0} a_k\lambda_k^{1/2}x_k$$

has entries of the same sign as those of $X_1$ on $(E_1 \cup F_1)^c$. Doing the same construction for $X_2$, we obtain an index $n_0$ and a Gaussian vector

$$\tilde{X}_2 = \sum_{k=1}^{n_0} b_k\big(\lambda'_k\big)^{1/2}y_k$$

with entries of the same sign as those of $X_2$ outside an event $E_2 \cup F_2$.

Last, we estimate the total variation distance between the joint distribution of $(\tilde{X}_1, \tilde{X}_2)$ and that of independent variables with the same marginals (which is what one would obtain starting from Y rather than X). This is the same as the total variation distance between the joint law of $Z := (a_1, \dots, a_{m_0}, b_1, \dots, b_{n_0})$ and that of independent normal variables. The covariance matrix $\Sigma_Z$ of Z is of block form, with two identity diagonal blocks and two off-diagonal blocks with entries of absolute value at most $\eta\sqrt{mn}/\lambda$ (this bound follows by Cauchy-Schwarz). Using the Pinsker inequality and standard bounds for Gaussian vectors:

$$\begin{aligned} d_{TV}\big(\mathcal{N}(0, \Sigma_Z), \mathcal{N}(0, I_{m_0+n_0})\big) &\leqslant \sqrt{\frac{1}{2}D_{KL}\big(\mathcal{N}(0, \Sigma_Z)\|\mathcal{N}(0, I_{m_0+n_0})\big)} \\ &= \frac{1}{2}\sqrt{|\log\det\Sigma_Z|}, \end{aligned}$$

where $D_{KL}$ denotes the Kullback-Leibler divergence. By the Gershgorin circle theorem the eigenvalues of $\Sigma_Z$ are within distance $(m+n)\eta\sqrt{mn}/\lambda$ of 1, so

$$|\log\det\Sigma_Z| \leqslant -(m+n)\log\big(1-(m+n)\eta\sqrt{mn}/\lambda\big) \leqslant \frac{2(m+n)^3\eta}{\lambda}$$

for $(m+n)^2\eta/\lambda$ small enough.

Doing the same construction starting from Y and putting everything together, we obtain

$$\begin{aligned} d_{TV}(\mu_X, \mu_Y) &\leqslant 2P[E_1 \cup E_2 \cup F_1 \cup F_2] + d_{TV}\big(\mathcal{N}(0, \Sigma_Z), \mathcal{N}(0, I_{m_0+n_0})\big) \\ &\leqslant 2(m+n)\varepsilon + \frac{8\lambda(m+n)^3}{\varepsilon^2} + (m+n)^{3/2}\sqrt{\frac{2\eta}{\lambda}}. \end{aligned}$$

Choosing $\lambda = \varepsilon^3/(4(m+n)^2)$ gives

$$d_{TV}(\mu_X, \mu_Y) \leqslant 4(m+n)\varepsilon + 2\sqrt{2}(m+n)^{5/2}\sqrt{\frac{\eta}{\varepsilon^3}}$$

and finally setting $\varepsilon = [(m+n)^3\eta/2]^{1/5}$ we obtain

$$d_{\mathrm{TV}}(\mu_{\mathrm{X}}, \mu_{\mathrm{Y}}) \leqslant 2^{14/5}(m+n)^{8/5}\eta^{1/5}$$

thus proving the announced inequality with $\mathrm{C} = 2^{14/5}$. To validate the logarithm estimate above, notice that

$$\frac{(m+n)^2\eta}{\lambda} = \mathcal{O}\big((m+n)^{11/5}\eta^{2/5}\big) = \big[(m+n)^{8/5}\eta^{1/5}\big]^{11/8}\mathcal{O}\big(\eta^{5/40}\big)$$

so the only case where it is not small is when our bound is at least of order 1, in which case the statement of the theorem is vacuous (but true). □

*Proof of Proposition 4.1.* — Define $\mathrm{X} \in \mathbf{R}^{\mathrm{N}}$ the following random Gaussian vector. Denote by $\{x_i\}_{i\in\{1,\dots,m\}}$ the elements of $\mathrm{S} \cap \mathcal{T}$, and by $\{y_j\}_{i\in\{1,\dots,n\}}$ the elements of $\mathrm{T} \cap \mathcal{T}$. Define

$$\mathrm{X} = \big(f(x_1), \dots, f(x_m), f(y_1), \dots, f(y_n)\big) \in \mathbf{R}^{m+n}.$$

Using the notations of Theorem 4.3, the coefficients of $\Sigma_{12}$ are the $\mathrm{K}(x_i - y_j)$. Moreover the diagonal entries of $\Sigma_{\mathrm{X}}$ are equal to $\mathrm{K}(0) = 1$. Since there exists a universal constant C such that $(m+n) \leqslant \mathrm{C}a_{\mathcal{T}}\mathrm{A}rea(\mathrm{S} \cup \mathrm{T})$, the first assertion is a direct consequence of Theorem 4.3. □

*Estimates.* — In this paragraph, we obtain some bounds for the parameters $c_0(\Omega(f, \mathcal{T}))$ of Definition 2.1 and $\alpha(\Omega(f, \mathcal{T}), \cdot)$ of Lemma 2.7.

*Lemma* **4.4.** — *Let $f$ be a $\mathrm{C}^0$ stationary Gaussian field on $\mathbf{R}^2$ satisfying the condition (3.3.) (symmetry), and $\mathcal{T}$ be a lattice satisfying the conditions of Definition 3.3. Then $\Omega(f, \mathcal{T})$ satisfies condition 2. (symmetry) and condition 3. (percolation through squares) of Definition 2.1. More precisely,*

$$\forall s \in \mathbf{N}^*, \quad f_s\big(\Omega(f, \mathcal{T}), 1\big) = c_0\big(\Omega(f, \mathcal{T})\big) = 1/2.$$

*Proof.* — Since $\mathcal{T}$ and the covariance function of $f$ are invariant under the symmetries of the axes and $\pi/2$-rotation, the process $\Omega(f, \mathcal{T})$ satisfies condition 2. of Definition 2.1. Moreover, since $\mathcal{T}$ is a triangulation, since the vertices lie on $(\frac{1}{\mathrm{N}}\mathbf{Z})^2$, on any square $\mathrm{B}_s$ with $s \in \mathbf{N}^*$, either there is an horizontal black crossing in $\mathcal{V}$, that is an arc $c$ in $\mathcal{T}$ joining $\{-s\} \times [-s, s]$ to $\{s\} \times [-s, s]$ in $\mathrm{B}_s$ such that $f_{|c\cap\mathcal{V}} > 0$, or there is a vertical negative arc. Since the coefficients $(a_{ij})_{ij}$ are centered Gaussian, since the square, the measure and $\mathcal{T}$ are invariant under a $\pi/2$-rotation, both events happen with the same probability, so that they both have probability equal to $1/2$. By invariance under translation, the probability of a black crossing in any square is $1/2$. In other words, for any $s \in \mathbf{N}^*, f_s(\Omega(f, \mathcal{T}), 1) = 1/2$. □

*Remark* **4.5.** — Note that Lemma 4.4 does not require the polynomial decay of the correlation nor its positivity, and holds in particular for the random wave model given by (1.7).

In the proof of Theorem 1.1 we will need an estimate of the parameters $t_\nu(\Omega(f, \varepsilon\mathcal{T}))$ for our discretized processes, which implies in particular an estimate for $\alpha(s(\Omega(f, \varepsilon\mathcal{T}))$, where $\alpha(\Omega, \cdot)$ is defined in Lemma 2.7 and $s(\Omega(f, \varepsilon\mathcal{T})$ by (2.6).

*Lemma* **4.6.** — *Suppose that $f$ is a $C^0$ stationary Gaussian field defined on $\mathbf{R}^2$ and satisfying conditions (3.2) and (3.3). Then, there exist $a, b > 0$, such that after changing the universal polynomial $Q_1$ in Lemma 2.7 into $aQ_1$, the following holds: for any periodic lattice $\mathcal{T}$ satisfying all of the conditions of Definition 3.3,*

$$\forall s \in \mathbf{N}^*, \quad \alpha\big(\Omega(f, \mathcal{T}), s\big) \geqslant b.$$

*Proof.* — Recall that $\alpha(\Omega, s)$ is a constant that must satisfy the two conditions **(P1)** and **(P2)** of Lemma 2.7. Denote by $b^+$ and $b^-$ the two squares

$$b^\pm = [\pm s/2 - \lambda/2, \pm s/2 + \lambda/2] \times [-\lambda/2, \lambda/2]$$

and by $\mathcal{A}^\pm$ the black-increasing events $\mathcal{A}^\pm = \{f_{|b^\pm \cap \mathcal{V}} > 0\}$, where $\mathcal{V}$ is the set of vertices of $\mathcal{T}$. By Lemma 3.12 and the FKG inequality given by Lemma 3.10, there exists $\tau > 0$ depending only on $f$ such that $\forall s \geqslant 1, \mathbf{P}[\mathcal{A}^+ \cap \mathcal{A}^-] \geqslant \tau^2$. Assume that $\alpha(\Omega(f, \mathcal{T}), s) < \lambda/4$. Then we have

$$\big\{\mathcal{X}_s\big(\alpha\big(\Omega(f, \mathcal{T}), s\big)\big)\big\} \cap \mathcal{A}^+ \cap \mathcal{A}^- \subset \big\{\mathcal{X}_s(\lambda/4)\big\}$$

so that, again by FKG inequality and (**P1**), $\mathbf{P}[\mathcal{X}_s(\lambda/4)] \geqslant Q_1(c_0)\tau^2$, where $Q_1$ is defined in Lemma 2.7, so that $\lambda/4$ satisfies **(P1)** after replacing $Q_1$ by $\tau^2 Q_1$. Moreover,

$$\begin{aligned}\mathbf{P}\big[\mathcal{H}_s(0, \lambda/4)\big] - \mathbf{P}\big[\mathcal{H}_s(\lambda/4, s/2)\big] \geqslant{} & \mathbf{P}\big[\mathcal{H}_s\big(0, \alpha\big(\Omega(f, \mathcal{T}), s\big)\big)\big] \\ & - \mathbf{P}\big[\mathcal{H}_s\big(\alpha\big(\Omega(f, \mathcal{T}), s\big), s/2\big)\big] \\ \geqslant{} & c_0/4,\end{aligned}$$

so that $\lambda/4$ satisfies condition **(P2)**. In conclusion, we can replace $\alpha(\Omega(f, \mathcal{T}), s)$ by $\lambda/4$. $\square$

The following Theorem 4.7 provides a large family of correlated percolations on lattices, and has its own interest. Moreover, it provides a bound for the main parameter $t_\nu(\Omega)$ which will be used in the proof of the main Theorem 1.1. Recall that $\gamma(\nu) = 1 + \log_{4/(3+2\nu)}(3/2 + \nu)$, see (2.8).

*Theorem* **4.7.** — *Let $f$ be a $C^0$ stationary Gaussian field satisfying the conditions (3.2) (positivity), (3.3) (symmetry) and (3.4) (polynomial decay) for $\alpha > 16$. Then, for any $\nu \in ]0, 1/2[$ and any $\theta \in ]0, \alpha - 16[$, there exists a constant $C_{\theta,\nu} > 0$ depending only on $(\theta, \nu)$ and the parameters of condition (3.4), such that for any periodic lattice $\mathcal{T}$ satisfying the conditions of Definition 3.3, the process $\Omega(f, \mathcal{T})$ satisfies the conditions of Theorem 2.2, with*

$$t_\nu\big(\Omega(f, \mathcal{T})\big) \leqslant C_{\theta,\nu} a_{\mathcal{T}}^{\frac{8\gamma(\nu)}{\alpha-16-\theta}}, \tag{4.2}$$

*where $a_{\mathcal{T}}$ denotes the number of vertices of $\mathcal{T}$ per unit square. Moreover, for any such lattice $\mathcal{T}$, for any $\rho \geqslant 1$, there exists $c > 0$,*

$$\forall s \in \mathbf{N}^*, \quad f_s\big(\Omega(f, \mathcal{T}), \rho\big) \geqslant c.$$

*Proof.* — For any lattice $\mathcal{T}$ satisfying the hypotheses, by Lemma 3.10 and Lemma 4.4, $\Omega(f, \mathcal{T})$ satisfies all of the conditions of Definition 2.1, so that Theorem 2.2 applies. By Lemma 2.9 and the existence of the universal function $Q_3$, by Lemma 4.4 and the existence of the universal parameter $c_0(\Omega(f, \mathcal{T})) = 1/2$, by the definition of $s(\Omega)$ given by (2.6) and by Corollary 4.2, for any $\theta \in ]0, \alpha - 16[$, there exists a constant $C_\theta > 0$ depending only on $\theta$ and the parameters $\alpha, \beta$ of the polynomial decay (3.4) such that for any $\mathcal{T}$,

$$s\big(\Omega(f, \mathcal{T})\big) \leqslant C_\theta a_{\mathcal{T}}^{\frac{8}{\alpha-16-\theta}}. \tag{4.3}$$

Moreover by Lemma 4.6, there exists $a > 0$, such that for any lattice $\mathcal{T}$,

$$\alpha\Big(s\big(\Omega(f, \mathcal{T})\big)\Big) \geqslant a,$$

so that by (2.9), (4.3) and the definition (2.10) of $s_\nu(\Omega)$, the parameter $t_\nu(\Omega(f, \mathcal{T}))$ satisfies the upper bound (4.2).

Now by Remark 3.15, there exists $a > 0$, such that

$$\mathbf{P}(f_{|B_{\rho t_\nu(\Omega(f,\mathcal{T}))} \cap \mathcal{V}} > 0) \geqslant a,$$

hence the second result. $\square$

*Remark* **4.8.** — For random sums of Gaussian waves, the stationary correlation function is given by (1.8), so that $\forall x \in \mathbf{R}^2$, $|K(x)| \leqslant \|x\|^{-1/2}$. Hence, a priori Theorem 4.7 does not apply in this case. Notice that the FKG inequality also fails to hold in that model, so it is beyond the techniques that we develop here for several reasons.

**4.2.** *Proof of Theorem 1.1.* — This section is devoted to the proof of the main result of this paper:

*Theorem* **4.9.** — *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a $C^4$ random Gaussian field satisfying conditions (3.1), (3.2), (3.3), (3.4) for $\alpha > 144 + 128\log_{4/3}(3/2)$ and (3.5). Let $\Gamma = (U, \gamma, \gamma')$ be a quad, that is a triple given by a smooth bounded open connected set $U \subset \mathbf{R}^2$, and two disjoint compact smooth arcs $\gamma$ and $\gamma'$ in $\partial U$. Then, there exists a positive constant $c$ such that:*

1. *for any positive $s$, with probability at least $c$ there exists a connected component of $\{x \in \bar{U}, f(sx) > 0\}$ intersecting both $\gamma$ and $\gamma'$;*
2. *there exists $s_0 > 0$, such that for any $s \geqslant s_0$, with probability at least $c$ there exists a connected component of $\{x \in \bar{U}, f(sx) = 0\}$ intersecting both $\gamma$ and $\gamma'$.*

*Remark* **4.10.** — The same result holds if we replace the quad $(U, \gamma, \gamma')$ by any smooth topological ring A and the existence of a positive or nodal circuit in A generating $\pi_1(A)$.

*Proof of Theorem 1.1.* — By Lemma 3.6, the correlation function given by (1.3) for random elements of $\mathcal{W}$ (or the Wiener space $\mathcal{W}(\mathcal{F})$ associated to the Bargmann-Fock space) satisfies the conditions of Theorem 4.9. □

Theorem 4.9 will be an easy consequence of the following main proposition:

*Proposition* **4.11.** — *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a random Gaussian field satisfying the hypotheses of Theorem 4.9. Let $(R_i)_{i\in I}$ and $(S_j)_{j\in J}$ be two finite families of horizontal or vertical rectangles, such that: $\forall (i,j) \in I \times J, R_i \cap S_j = \emptyset$. Then there exists $c > 0$ such that*

1. *for any positive $s$, the following event $\Omega_R(s)$ happen with probability at least $c$: for every $i \in I$, there exists a positive path crossing $sR_i$ in its length;*
2. *for any positive $s$, the following event $\Omega_S(s)$ happen with probability at least $c$: for every $j \in J$, there exists a negative path crossing $sS_j$ in the length;*
3. *there exists $s_0 > 0$, such that for any $s \geqslant s_0$, $\mathbf{P}[\Omega_R(s) \cap \Omega_S(s)] \geqslant c$.*

*Proof of Theorem 4.9.* — Let $U \subset \mathbf{R}^2$ be a connected smooth bounded open set and $\gamma, \gamma' \subset \partial U$ two disjoint connected arcs in the boundary of U. First, it is clear that there exist a pair of rectangles $R_+$ and $R_-$ together with a finite family of horizontal or vertical rectangles $(R_i \subset U)_{i\in I}$ (resp. $S_+$, $S_-$ and $(S_j \subset U)_{i\in J}$) such that, writing $\tilde{I} = I \cup \{\pm\}$ and $\tilde{J} = J \cup \{\pm\}$,

- $\forall (i,j) \in (\tilde{I} \times \tilde{J}), R_i \cap S_j = \emptyset$,
- the arc $\gamma$ traverses $R_-$ and $S_-$ through their shortest width,
- the arc $\gamma'$ traverses $R_+$ and $S_+$ through their shortest width,
- if for all $i \in \tilde{I}$, there exists a path $\gamma_i$ crossing $R_i$ in its length, then $\bigcup_{i\in\tilde{I}} \gamma_i$ is connected.
- if for all $j \in \tilde{J}$, there exists a path $\gamma'_j$ crossing $S_j$ in its length, $\bigcup_{j\in\tilde{J}} \gamma'_i$ is connected.

By Proposition 4.11 for any $s > 0$, with probability $c$ there exists a positive path, *i.e.* a connected component of $\{x \in \mathbf{R}^2, f(x) > 0\}$, (resp. a negative path) in $s(\bigcup_{i\in\tilde{\mathrm{I}}} \mathrm{R}_i)$ (resp. $s(\bigcup_{j\in\tilde{\mathrm{J}}} \mathrm{S}_j)$), hence the first assertion.

By the second assertion of Proposition 4.11, there exists $s_0 > 0$, such that for $s \geqslant s_0$, both events happen with probability at least $c > 0$, so that a crossing nodal line appears between them with at least the same probability. $\square$

### 4.3. *Proof of Proposition 4.11.*

*Heuristics of the proof.* — As explained in the introduction, we need to use a family of intermediary processes which are percolations on lattices of smaller and smaller mesh as the rectangles to be crossed become larger. More precisely, fix $\mathcal{T}$ a periodic graph and $\mathcal{V}$ its set of vertices. For any $\varepsilon > 0$, we will consider the rescaled lattice $\varepsilon\mathcal{T}$, so that $\varepsilon\mathcal{V}$ denotes its associated set of vertices and $\varepsilon\mathcal{E}$ its set of edges. Note that

$$\forall \varepsilon > 0, \quad a_{\varepsilon\mathcal{T}} = a_{\mathcal{T}}\varepsilon^{-2}, \tag{4.4}$$

where $a_{\mathcal{T}}$ is the mean number of vertices of $\mathcal{V}$ per unit square, see Section 4.1. For any $\varepsilon > 0$ and any random field $f$ on $\mathbf{R}^2$, the restriction of $f$ to $\varepsilon\mathcal{V}$ define the random coloring discrete process

$$\Omega_\varepsilon = \Omega(f, \varepsilon\mathcal{T}) \tag{4.5}$$

given by Definition 3.1. We assume for a moment that Theorem 1.6 is true (its proof is postponed to the last Sect. 5). Le $\mathrm{R} > 0$ be such that $\mathrm{B_R}$ contains all of the rectangles $(\mathrm{R}_i)_{i\in\mathrm{I}}$ and $(\mathrm{S}_j)_{j\in\mathrm{J}}$. For any $\sigma > 0$, Theorem 1.6 gives a size $\varepsilon(\sigma)$ such that with a large probability percolation for the process $\Omega_{\varepsilon(\sigma)}$ in a rectangle in $\mathrm{B}_{\mathrm{R}\sigma}$ is equivalent to continuous percolation for the process $\Omega(f)$, see Definition 3.1. Then, by Tassion's Theorem, if $\mathcal{T}$ satisfies the conditions of Definition 3.3, there is a size $t_\nu(\Omega_{\varepsilon(\sigma)})$, such that with probability at least $c > 0$, percolation happens for $\Omega_{\varepsilon(\sigma)}$ in the length of given rectangles of sizes larger than $t_\nu(\Omega_{\varepsilon(\sigma)})$. Since the correlation function of $f$ decreases polynomially for a sufficiently high degree, this size can be chosen to be smaller than $\sigma$, so that we get percolation for the process $\Omega(f)$ for the rectangles of size $\sigma$. If the two families of rectangles are far from each other, again by the polynomial decay of the correlation, positive and negative crossings happen simultaneously.

*Proof of Proposition 4.11.* — Fix $\mathcal{T}$ any lattice satisfying the conditions of Definition 3.3. We begin to prove uniform percolation on one unique rectangle, but for the family of discretized percolations defined by $\Omega_\varepsilon$, see (4.5), where $\varepsilon \in (\mathbf{N}^*)^{-1}$ will depend on the size $s$ of the rectangle. By Lemma 3.10, for any $\varepsilon > 0$ the process $\Omega_\varepsilon$ satisfies condition 1. of Definition 2.1, and by Lemma 4.4, for any $\varepsilon \in (\mathbf{N}^*)^{-1}$ it satisfies the two other conditions 2. and 3., with $c_0(\Omega_\varepsilon) = 1/2$. Hence, the hypotheses of Theorem 2.2 are fulfilled. This implies, if $\rho \geqslant 1$ and $\nu \in ]0, 1/2[$ are fixed, that

$$\forall \varepsilon \in \left(\mathbf{N}^*\right)^{-1},\ \forall s \in \mathbf{N}^*,\ s \geqslant t_\nu(\Omega_\varepsilon), \quad f_s(\Omega_\varepsilon, \rho) \geqslant \mathrm{P}_\nu(\rho, 1/2),$$

where $\mathrm{P}_\nu$ is the universal function defined in Theorem 2.2. By Theorem 4.7 and (4.4), if

$$\gamma(\nu) = 1 + \log_{4/(3+2\nu)}(3/2+\nu),$$

see (2.8), for any $\theta \in ]0, \alpha - 16[$, there exists $\mathrm{C}_{\theta,\nu} > 0$ such that

$$\forall \varepsilon \in \left(\mathbf{N}^*\right)^{-1}, \quad t_\nu(\Omega_\varepsilon) \leqslant \mathrm{C}_{\theta,\nu} a_{\mathcal{T}}^{\frac{8\gamma(\nu)}{\alpha-16-\theta}} \varepsilon^{-\frac{16\gamma(\nu)}{\alpha-16-\theta}}. \tag{4.6}$$

Since $\alpha > 16 + 128\gamma(0)$, so that $\alpha - 144 > 0$ since $\gamma(0) > 1$, we can choose $\nu \in ]0, 1/2[$ and $\theta \in ]0, \alpha - 16[$ such that

$$\eta = \inf\left\{\frac{\alpha - 16 - \theta}{\gamma(\nu)} - 128, \alpha - 144\right\} > 0. \tag{4.7}$$

Choose $\mathrm{R} > 0$ such that

$$\left(\bigcup_{i\in \mathrm{I}} \mathrm{R}_i \bigcup_{j\in \mathrm{J}} \mathrm{S}_j\right) \subset \mathrm{B}_\mathrm{R}$$

and define for any $\sigma \geqslant 2$

$$\varepsilon(\sigma) = \left(\left\lfloor (\mathrm{R}\sigma)^{8+\frac{\eta}{32}} \right\rfloor + 1\right)^{-1} \in \left(\mathbf{N}^*\right)^{-1}. \tag{4.8}$$

This choice is required by Theorem 1.6. By (4.6) and (4.7), there exists a constant $\mathrm{C} > 0$ such that

$$\forall \sigma \geqslant 2, \quad t_\nu(\Omega_{\varepsilon(\sigma)}) \leqslant \mathrm{C}\sigma^{\frac{128+\eta/2}{\alpha-16-\theta}\gamma(\nu)} \leqslant \mathrm{C}\sigma^{\frac{128+\eta/2}{128+\eta}} = o(\sigma).$$

Hence, there exists $s_1 \geqslant 2$ not depending on $\rho \geqslant 1$ so that by Theorem 2.2,

$$\forall s \in \mathbf{N}^*,\ s \geqslant s_1, \quad f_s(\Omega_{\varepsilon(s)}, \rho) \geqslant \mathrm{P}_\nu(\rho, 1/2). \tag{4.9}$$

Now we turn to the simultaneous discrete percolation through our two sets of rectangles. For this, for any $i \in \mathrm{I}$ (resp. $j \in \mathrm{J}$), denote by $\rho_i$ (resp. $\rho'_j$) the quotient of the length and the width of $\mathrm{R}_i$ (resp. $\mathrm{S}_j$). If

$$c := \min_{(i,j)\in \mathrm{I}\times\mathrm{J}} \left(\mathrm{P}_\nu(\rho_i, 1/2), \mathrm{P}_\nu(\rho_j, 1/2)\right) > 0,$$

then by (4.9),

$$\forall s \in \mathbf{N}^*,\ s \geqslant s_1, \quad \min_{(i,j)\in \mathrm{I}\times\mathrm{J}} \left(f_s(\Omega_{\varepsilon(s)}, \rho_i), f_s\big(\Omega_{\varepsilon(s)}, \rho'_j\big)\right) \geqslant c. \tag{4.10}$$

For $s \geqslant 1$, define $\Omega_+(s)$ the event that for every $i \in \mathrm{I}$, there exists a positive path crossing for $\Omega_{\varepsilon(s)}$ in the length of $s\mathrm{R}_i$. We denote by $\Omega_-(s)$ the analogous event, where we change the sign and the $\mathrm{R}_i$'s are replaced by the $\mathrm{S}_j$'s. By the FKG inequalities given by Lemma 3.10,

$$(\mathbf{4.11}) \qquad \forall s \in \mathbf{N}^*,\ s \geqslant s_1, \quad \mathbf{P}\big[\Omega_+(s)\big] \geqslant c^{|\mathrm{I}|} \text{ and } \mathbf{P}\big[\Omega_-(s)\big] \geqslant c^{|\mathrm{J}|}.$$

Now, we want to obtain percolation for the continuous process $\Omega(f)$. For this, first fix

$$(\mathbf{4.12}) \qquad \delta = c^{|\mathrm{I}|+|\mathrm{J}|} \leqslant \min\big(c^{|\mathrm{I}|}, c^{|\mathrm{J}|}\big).$$

By Theorem 1.6 and (4.8), there is $s_2 := s_2(\delta/4, \eta)$ such that for $s \geqslant s_2$, with probability at least $1-\delta/4$, any positive crossing in the lattice $\varepsilon(s)\mathcal{T} \cap \mathrm{B}_{\mathrm{R}s}$ will produce a continuous positive crossing for $\Omega(f)$, as well as for the negative crossings. Hence by (4.11) and (4.12),

$$\forall s \in \mathbf{N}^*, \quad s \geqslant \max(s_1, s_2), \quad \min\big(\mathbf{P}\big[\Omega_{\mathrm{R}}(s)\big], \mathbf{P}\big[\Omega_{\mathrm{R}}(s)\big]\big) \geqslant 3\delta/4,$$

where $\Omega_{\mathrm{R}}$ and $\Omega_{\mathrm{S}}$ are defined in Proposition 4.11. Moreover, by Lemma 3.14 applied to $s = \mathrm{R}\max(s_1, s_2)$, there exists $a > 0$, such that for any rectangle in $\mathrm{B}_{\mathrm{R}\max(s_1,s_2)}$, with probability at least $a$, there exists a (trivial) positive (resp. negative) crossing of this rectangle. Summarizing, and using again FKG inequality for the latter elementary percolations,

$$(\mathbf{4.13}) \qquad \forall s \in \mathbf{N}^* \cup \big[0, \max(s_1, s_2)\big], \quad \min\big(\mathbf{P}\big[\Omega_{\mathrm{R}}(s)\big], \mathbf{P}\big[\Omega_{\mathrm{S}}(s)\big]\big) \geqslant \min\big(3\delta/4, a^{|\mathrm{I}|}, a^{|\mathrm{J}|}\big).$$

In order to remove the integer condition, note that we can translate our lattice $\mathcal{T}$ by any vector in a fundamental square of $\mathcal{T}$. Since the Gaussian field is stationary, the probabilities obtained for the translated rectangles (again with integer sizes) the associated estimates (4.13) are the same. Remark 2.6 concludes.

For the second assertion, we prove that the two events $\Omega_+(s)$ and $\Omega_-(s)$ defined below happen simultaneously with uniform positive probability. By (4.11) it is enough to prove that these two events are almost independent. Let $\mathrm{M}, \mu > 0$ be such that

$$\mathrm{M} = \max\Big(\mathrm{area}\Big(\bigcup_i \mathrm{R}_i\Big), \mathrm{area}\Big(\bigcup_j \mathrm{S}_j\Big)\Big) \quad \text{and} \quad \mu = \mathrm{dist}\Big(\bigcup_i \mathrm{R}_i, \bigcup_j \mathrm{S}_j\Big).$$

Corollary 4.2 and (4.7) show that there exist $\mathrm{C}', \mathrm{C}'' > 0$, such that for any $s \geqslant 1$, with probability at least

$$1 - \mathrm{C}'\big(\mathrm{M} a_{\mathcal{T}} \varepsilon(s)^{-2} s^2\big)^{8/5} (\mu s)^{-\frac{\alpha}{5}} = 1 - \mathrm{C}'' s^{\frac{128+16-\alpha+\eta/2}{5}} \geqslant 1 - s^{-\frac{\eta}{10}},$$

the signs on $s(\bigcup_i \mathrm{R}_i) \cap \varepsilon(s)\mathcal{V}$ and $s(\bigcup_j \mathrm{S}_j) \cap \varepsilon(s)\mathcal{V}$ can be coupled with the realization of a pair of independent colorings, where $\varepsilon(s)$ is defined by (4.8). Hence, there exists $s_3 \geqslant 1$ such that this probability is larger than $1-\delta/4$ for $s \geqslant s_3$, where $\delta$ is defined by (4.12). Consequently, by (4.11),

$$\forall s \in \mathbf{N}^*, \quad s \geqslant \max(s_1, s_3),$$

$$\mathbf{P}\big[\Omega_+(s)\cap\Omega_-(s)\big]\geqslant \mathbf{P}\big[\Omega_+(s)\big]\mathbf{P}\big[\Omega_-(s)\big]-\delta/4\geqslant 3\delta/4,$$

and hence by the definition of $s_2$ above,

$$\forall s\in\mathbf{N}^*,\ \ s\geqslant \max(s_1,s_2,s_3),\qquad \mathbf{P}\big[\Omega_{\mathrm{R}}(s)\cap\Omega_{\mathrm{S}}(s)\big]\geqslant\delta/2.$$

We can remove the integer condition as above, so that the second assertion of Proposition 4.11 is proved. □

*Remark* **4.12.** — There are at least three sources of lowering the degree $144+128\log_{4/3}(3/2)$, namely in [44, Lemma 2.2], in the proof of Theorem 4.3 and in the size of the box given by the quantitative implicit theorem given by Corollary A.3.

## 5. Proof of Theorem 1.6

The goal of this section is to prove the main Theorem 1.6. This theorem states that for any stationary Gaussian field satisfying condition (3.5), with an arbitrary large probability, for any $s$ large enough and $\varepsilon$ small enough (depending on $s$), any random nodal line in $\mathrm{B}_s$ will cross the edges of $\varepsilon\mathcal{T}$ at most once. In this case, for any pair of adjacent vertices with same sign, $f$ does not vanish on the associated edge and has the sign of the vertices. We recall the result.

*Theorem* **1.6.** — *Let $f$ be a $\mathrm{C}^4$ random stationary Gaussian field on $\mathbf{R}^2$ satisfying the non-degeneracy condition (3.5), $\mathcal{T}$ be a periodic lattice, and $\mathcal{E}$ be its set of edges. Fix $\eta>0$. Then there exists $s(\eta)>0$, $\mathrm{C}(\eta)>0$ and $\alpha(\eta)>0$ such that for every $s\geqslant s(\eta)$, every $\delta>0$ and every $\varepsilon\leqslant \mathrm{C}(\eta)\delta^{\alpha(\eta)}s^{-8-\eta}$, with probability at least $1-\delta$ the following event happens:*

$$\forall e\in\varepsilon\mathcal{E}\cap\mathrm{B}_s,\qquad \#\big(e\cap f^{-1}(0)\big)\leqslant 1. \tag{5.1}$$

*Remark* **5.1.** — Under the event 5.1, any black percolation in $\mathrm{B}_s$ associated to $\Omega(f,\varepsilon\mathcal{T})$ will provide an associated continuous path over which $f$ is positive, hence giving a continuous percolation on $\mathrm{B}_s$ associated to $\Omega(f)$. This theorem has its own interest since it gives a very general link between continuous percolation and discrete percolation.

**5.1.** *Quantitative bounds.* — First, we prove some results on Gaussian fields that will be useful when in the second part we will add a lattice. Recall that $\mathrm{B}_s=[-s,s]^n$.

*Lemma* **5.2.** — *Let $f:\mathbf{R}^n\to\mathbf{R}$ be a $\mathrm{C}^4$ stationary Gaussian field. Fix $p>0$. Then there exists $\mathrm{C}_p>0$, such that for every $s\geqslant 2$,*

$$\mathbf{E}\big(\|f\|^p_{\mathrm{C}^2(\mathrm{B}_s)}\big)\leqslant \mathrm{C}_p(\log s)^{p/2}.$$

*Proof.* — We begin with $p=1$. We use a classical result by Dudley [1, Theorem 1.3.3]. For this, define the semi-distance on $\mathbf{R}^n$ by

$$\forall x,y\in\mathbf{R}^n,\quad d^2(x,y)=\mathbf{E}\Big(\big(f(x)-f(y)\big)^2\Big).$$

In our case, $d(x,y)=\sqrt{2}\sqrt{1-\mathrm{K}(x-y)}$. For any $\eta>0$, denote by $\mathrm{N}_s(\varepsilon)$ the number of balls for $d$ of size $\varepsilon$ needed to cover $\mathrm{B}_s=[-s,s]^n$. Then by [1, Theorem 1.3.3] (see also for instance [19]), there exists a universal constant $\mathrm{C}>0$ such that

$$\forall s>0,\quad \mathbf{E}(\|f\|_{\mathrm{L}^\infty(\mathrm{B}_s)})\leqslant \mathrm{C}\int_0^{\mathrm{diam}(\mathrm{B}_s)/2}\sqrt{\log \mathrm{N}_s(\varepsilon)}\,d\varepsilon.$$

Here, $\mathrm{diam}(\mathrm{B}_s)$ is the diameter of $\mathrm{B}_s$ for the pseudo-metric $d$. Since $d\leqslant\sqrt{2}$, we get $\mathrm{diam}(\mathrm{B}_s)\leqslant\sqrt{2}$. Moreover, since K is $\mathrm{C}^2$ and $d\mathrm{K}(0)=0$ by Remark 3.7, by Taylor there exists $c>0$ such that

$$\forall (x,y)\in\big(\mathbf{R}^n\big)^2,\quad d(x,y)\leqslant c\|x-y\|.$$

Hence,

$$\exists c'>0,\ \forall\varepsilon>0,\ \forall s\geqslant 2,\quad 1\leqslant \mathrm{N}_s(\varepsilon)\leqslant c'\left(\frac{s}{\varepsilon}\right)^n,$$

so that there is $c_0>0$, for all $s\geqslant 2$,

$$\mathbf{E}\big(\|f\|_{\mathrm{L}^\infty(\mathrm{B}_s)}\big)\leqslant c_0\sqrt{\log s}. \tag{5.2}$$

Now, for any $i\in\{1,\dots,n\}$, $\frac{\partial f}{\partial x_i}$ is a Gaussian field with covariance function

$$\forall (x,y)\in\big(\mathbf{R}^n\big)^2,\quad \mathbf{E}\big(\partial_{x_i}f(x)\partial_{x_i}f(y)\big)=\partial^2_{x_i,y_i}\big(\mathrm{K}(x-y)\big)=-\big(\partial^2_{x_i^2}\mathrm{K}\big)(x-y).$$

Consequently, the pseudo-metric $d_i$ associated to $\partial_{x_i}f$ equals

$$\forall (x,y)\in\big(\mathbf{R}^n\big)^2,\quad d_i(x,y)=\sqrt{2}\sqrt{-\partial^2_{x_i^2}\mathrm{K}(0)+\big(\partial^2_{x_i^2}\mathrm{K}\big)(x-y)}.$$

Then $\mathbf{R}^n$ has finite diameter and using again Taylor we get that $d_i$ is bounded above by the flat metric up to a multiplicative constant, so that as before there is a constant $c''>0$ such that $\mathrm{N}_s(\varepsilon)$ for this field is bounded by $c''(\frac{s}{\varepsilon})^n$. Hence, there exists $c_{i,1}>0$ such that for every $s\geqslant 2$, $\mathbf{E}(\|\partial_{x_i}f\|_{\mathrm{L}^\infty(\mathrm{B}_s)})\leqslant c_{i,1}\sqrt{\log s}$ and summing up for $i\in\{1,\dots,n\}$, we obtain the existence of $c_1>0$, such that

$$\forall s\geqslant 2,\quad \mathbf{E}(\|df\|_{\mathrm{L}^\infty(\mathrm{B}_s)})\leqslant c_1\sqrt{\log s}.$$

Similarly, since K is $\mathrm{C}^4$, there exists $c_2>0$ such that for every $s\geqslant 2$, $\mathbf{E}(\|d^2f\|_{\mathrm{L}^\infty(\mathrm{B}_s)})\leqslant c_2\sqrt{\log s}$. Adding the three estimates, we have proved the result for $p=1$.

Now, by the Borell-TIS inequality (see Theorem 2.1.1 in [1]),

$$\forall u \geqslant 0, \quad \mathbf{P}\Big[\|f\|_{L^\infty(B_s)} \geqslant \mathbf{E}\|f\|_{L^\infty(B_s)} + u\Big] \leqslant \exp\Big(-\frac{u^2}{2\sup_{x\in B_s} e(x,x)}\Big),$$

where $e(x,x) = K(0) = 1$, so that by (5.2),

$$\forall u \geqslant 0, \quad \mathbf{P}[\|f\|_{L^\infty(B_s)} \geqslant c_0\sqrt{\log s} + u] \leqslant \exp\Big(-\frac{1}{2}u^2\Big).$$

Hence, if $p > 0$, there exists $c''' > 0$ such that $\mathbf{E}(\|f\|^p_{L^\infty(B_s)})$ is bounded from above (after integration by parts) by

$$p(c_0\sqrt{\log s})^p + p\int_0^\infty (u + c_0\sqrt{\log s})^{p-1}\exp\Big(-\frac{1}{2}u^2\Big)du \leqslant c'''(\log s)^{p/2}.$$

The same estimate holds for the higher derivatives. □

When we add to the Gaussian field a lattice, we need to understand the scale $\varepsilon$ at which the zero set of $f$ is trivial, that is a local graph, on a large box of a given size, and to quantify the probability of this event and all of the involved signs. For this, we must show that $f$ is often quantitatively transverse when it vanishes, that is its derivative is bounded by a positive controlled uniform constant, and that the nodal line is not too curved. The last condition is ensured by Lemma 5.2. The following Lemma proves the first condition, and is a quantitative version of Lemma 7 in [35]. We follow here their proof, but keeping track of the constants depending on $s$ and therefore using Lemma 5.2.

*Lemma* **5.3** *(see [35]). — Let $f$ be a $C^4$ stationary Gaussian field on $\mathbf{R}^n$ satisfying condition (3.5). Fix $\eta > 0$. Then, there exists $\mu(\eta) > 0$ and $\kappa(\eta) > 0$, such that*

$$\forall s \geqslant 2,\ \forall \delta > 0, \quad \mathbf{P}\Big[\min_{x\in B_s}\max\big(|f(x)|, |df(x)|\big) < \mu(\eta)\delta^{\kappa(\eta)}s^{-n-\eta}\Big] \leqslant \delta.$$

As in the aforementioned article, the proof of Lemma 5.3 will need the following Lemma.

*Lemma* **5.4** *([35]). — Let $f$ be a $C^2$ stationary Gaussian field on $\mathbf{R}^n$ satisfying condition (3.5). For any $x \in \mathbf{R}^n$, define the random variable*

$$\Phi(x) = \frac{1}{|f(x)||df(x)|^n} \in \mathbf{R}_+ \cup \{+\infty\}.$$

*Then, for any $0 < \alpha < 1$, $\Phi(x)^\alpha$ is integrable and there exists $C_\alpha > 0$, such that for any $x \in \mathbf{R}^n$, $\mathbf{E}(\Phi(x)^\alpha) \leqslant C_\alpha$.*

*Proof.* — By invariance under translation, we only need to prove the existence of a finite $C_\alpha$ for a fixed $x$. Since $f$ is stationary, $\mathbf{E}(f(x)df(x)) = d_x^{1,0}e(x,x) = d\mathrm{K}(0)$ vanishes, see Remark 3.7, so that $f(x)$ and $df(x)$ are independent. For any $0 < \alpha < 1$, we then have

$$\mathbf{E}\big(\Phi(x)^\alpha\big) = \mathbf{E}\big(|f(x)|^{-\alpha}\big)\mathbf{E}\big(|df(x)|^{-n\alpha}\big).$$

By the coarea formula,

$$\mathbf{E}\big(|f(x)|^{-\alpha}\big) = \int_{y\in\mathbf{R}} \frac{1}{|y|^\alpha}\int_{f, f(x)=y} e(x,x)^{-1/2} d\mu_x(f)dy$$

which converges since $e(x,x) = \mathrm{K}(0) = 1$ and $\alpha < 1$. Here, $d\mu_x(f)$ denotes the Gaussian measure of the random variable $f(x)$. Likewise,

$$\mathbf{E}\big(|df(x)|^{-n\alpha}\big) = \int_{\mathrm{Y}\in\mathbf{R}^n} \frac{1}{|\mathrm{Y}|^{n\alpha}}\int_{f, df(x)=\mathrm{Y}} \left|\det d_{x,y}^{1,1}e(x,x)\right|^{-1/2} d\mu_x(f)d\mathrm{Y},$$

where $d\mathrm{Y}$ is the Lebesgue measure on $\mathbf{R}^n$. This integral converges since $\alpha < 1$ and $d_{x,y}^{1,1}e(x,x) = -d^2\mathrm{K}(0)$ which is non-degenerate by condition (3.5). □

*Proof of Lemma 5.3 (see Lemma 7 of [35]).* — Define

$$\mathrm{W} = 1 + \|f\|_{C^2(\mathrm{B}_s)},$$

where $\mathrm{B}_s = [-s,s]^n$. For any $0 < \tau < 1$, define

$$\mathrm{D}_\tau(s) = \big\{x \in \mathrm{B}_s, \max\big(|f(x)|, |df(x)|\big) \leqslant \tau\big\}$$

and $\Omega_\tau$ the event

$$\Omega_\tau = \big\{\mathrm{D}_\tau(s) \neq \emptyset\big\}.$$

Under $\Omega_\tau$, let $x \in \mathrm{D}_\tau(s)$. Then for every $y$ belonging to the round ball $\mathrm{B}(x,\tau)$,

$$|f(y)| \leqslant \tau + \|df\|_{\mathrm{L}^\infty(\mathrm{B}_s)}\tau \leqslant \mathrm{W}\tau.$$
$$|df(y)| \leqslant \tau + \|d^2f\|_{\mathrm{L}^\infty(\mathrm{B}_s)}\tau \leqslant \mathrm{W}\tau.$$

Consequently, there exists $\mathrm{C} > 0$ such that for $t > 0$,

$$\forall s \geqslant 2, \quad \int_{\mathrm{B}_s} \Phi^t(y)dy \geqslant \int_{\mathrm{B}(x,\tau)} \Phi^t(y)dy \geqslant \mathrm{C}\tau^{n-t(1+n)}\mathrm{W}^{-t(1+n)},$$

where $\Phi$ is given by Lemma 5.4. Therefore,

$$\forall s \geqslant 2, \quad \mathbf{E}\left(\mathrm{W}^{t(1+n)}\frac{1}{\mathrm{Vol}(\mathrm{B}_s)}\int_{\mathrm{B}_s}\Phi^t(y)dy\right) \geqslant \mathbf{P}[\Omega_\tau]\mathrm{C}\tau^{n-t(1+n)}\frac{1}{\mathrm{Vol}(\mathrm{B}_s)},$$

and if $\frac{1}{p}+\frac{1}{q}=1$, by Hölder inequality applied two times, for the variables $f$ and $y\in \mathrm{B}_s$, we obtain

$$\mathbf{P}[\Omega_\tau]\leqslant \frac{1}{\mathrm{C}}\mathrm{Vol}(\mathrm{B}_s)\tau^{t(1+n)-n}\Big(\mathbf{E}\big(\mathrm{W}^{t(1+n)p}\big)\Big)^{1/p}\bigg(\frac{1}{\mathrm{Vol}(\mathrm{B}_s)}\int_{\mathrm{B}_s}\mathbf{E}\big(\Phi^{tq}(y)\big)dy\bigg)^{1/q}.$$

By Lemma 5.2, there exists $\mathrm{C}'>0$ such that

$$\forall s\geqslant 2,\quad \mathbf{E}\big(\mathrm{W}^{t(1+n)p}\big)\leqslant \mathrm{C}'(\log s)^{\frac{p}{2}t(1+n)}.$$

Moreover, by Lemma 5.4, if $qt<1$, then $\mathbf{E}(\Phi^{tq}(\cdot))$ is uniformly bounded, and then there exists $\mathrm{C}_{q,t}>0$ such that

$$\forall\tau\in\,]0,1[,\ \forall s\geqslant 2,\quad \mathbf{P}(\Omega_\tau)\leqslant \mathrm{C}_{q,t}\,\mathrm{Vol}(\mathrm{B}_s)\tau^{t(1+n)-n}(\log s)^{\frac{t}{2}(1+n)}.$$

Hence, choosing $\tau$, for any $s\geqslant 2$, as

$$\tau=\delta^{(t(1+n)-n)^{-1}}\Big(\mathrm{C}_{q,t}\,\mathrm{Vol}(\mathrm{B}_s)(\log s)^{\frac{t}{2}(1+n)}\Big)^{-(t(1+n)-n)^{-1}},$$

we have $\mathbf{P}(\Omega_\tau)\leqslant\delta$. Now, we can choose $t<1$ close enough to 1, and then $q$ such that $1<q<1/t$, so that

$$\big(t(1+n)-n\big)^{-1}<1+\frac{\eta}{n+1}.$$

Hence, we get the result since $\mathrm{Vol}(\mathrm{B}_s)=(2s)^n$. □

**5.2.** *The lattice and the field.* — Recall that $\mathcal{T}$ denotes a periodic lattice on $\mathbf{R}^2$, and $\mathcal{E}$ denotes its set of edges. For any $\varepsilon>0$, we want to understand the link between the nodal line of the random $f$ and the signs on the edges of the rescaled lattice $\varepsilon\mathcal{T}$; in other words, between $\Omega(f)$ and $\Omega(f,\varepsilon\mathcal{T})$. So, we want to prove that double intersections with edges do not happen, with high probability. For this, let $v\in\mathbf{S}^1\subset\mathbf{R}^2$ be one of the directions of the edges, and let

- $\mathrm{H}(v,\varepsilon)$ be the subset of $\varepsilon\mathcal{E}$ defined by the edges of $\varepsilon\mathcal{E}$ parallel to $v$.

We want to estimate the mean number of nodal points $x$ which are $\theta$-close to $\mathrm{H}(v,\varepsilon)$ for $\theta>0$ and such that the tangent of $f^{-1}(0)$ at $x$ is parallel to $v$. Note that if $f^{-1}(0)$ is a local graph over an edge $e\in\varepsilon\mathcal{E}$, a double intersection of $f^{-1}(0)$ with $e$ gives birth to such a critical point.

*Lemma* **5.5.** — *Let $f$ be a $\mathrm{C}^2$ stationary Gaussian field on $\mathbf{R}^2$ satisfying condition (3.5). For every $\varepsilon,\theta,s>0$, every direction $v\in\mathbf{S}^1\subset\mathbf{R}^2$, define*

$$\mathrm{C}_\varepsilon(\theta,s,v,f)=\Big\{x\in\mathrm{B}_s,\ \mathrm{dist}\big(x,\mathrm{H}(v,\varepsilon)\big)\leqslant\theta,\ f(x)=0\ \textit{and}\ df(x)(v)=0\Big\}.\tag{5.3}$$

*Then, outside a set of vanishing measure,* $\mathrm{C}_\varepsilon(\theta, s, v, f)$ *is finite, and there exists* $\beta > 0$ *depending only on* $f$ *such that*

$$\forall \varepsilon, \theta, s > 0,\ \forall v \in \mathbf{S}^1, \quad \mathbf{E}\big(\#\mathrm{C}_\varepsilon(\theta, s, v, f)\big) \leqslant \beta \frac{s^2}{\varepsilon}\theta,$$

*so that* $\mathbf{P}[\mathrm{C}_\varepsilon(\theta, s, v, f) = \emptyset] \geqslant 1 - \beta\frac{s^2}{\varepsilon}\theta$.

*Remark* **5.6.** — In [25] and [21], the authors gave an explicit bound for the number of critical points of the restriction of a fixed Morse function (here the latter is a coordinate in the direction given by $v$) in order to bound the mean Betti numbers of random nodal hypersurfaces.

*Proof.* — Denote by $\mathrm{F} : \mathbf{R}^2 \to \mathbf{R}^2$ the Gaussian field defined by

$$\forall x \in \mathbf{R}^2, \quad \mathrm{F}(x) = \big(f, df(x)(v)\big).$$

This field is $\mathrm{C}^1$ and non-degenerate. Indeed, the associated covariance at $x$ is the matrix

$$\begin{pmatrix} e & d_x e(v) \\ d_y e(v) & d_x d_y e(v, v) \end{pmatrix}_{|(x,x)},$$

where $e(x, y) = \mathbf{E}(f(x), f(y)) = \mathrm{K}(x - y)$ denotes the covariance of $f$. This matrix is non-degenerate since the anti-diagonal terms vanish by Remark 3.7, since $e(x, x) = \mathrm{K}(0) = 1$ and since $d_x d_y e(x, x)(v, v) = -d^2\mathrm{K}(0)(v, v)$ is not degenerate by condition (3.5). Applying Theorem 6.2 of [5] where, with the notations of this monograph, $u = 0$ and

$$\mathrm{B} = \big\{x \in \mathrm{B}_s,\ \mathrm{dist}\big(x, \mathrm{H}(v, \varepsilon)\big) \leqslant \theta\big\},$$

we have

$$\mathbf{E}\big(\#\mathrm{C}_\varepsilon(\theta, s, v, f)\big) = \int_{\mathrm{B}} \mathbf{E}\big(|\det d\mathrm{Z}(x)| | \mathrm{Z}(x) = 0\big) p_{\mathrm{Z}(x)} dx,$$

where $p_{\mathrm{Z}(x)}$ denotes the density of Z.

Since B is compact, this integral is finite and depends continuously on $\mathrm{K}(0) = 1$ and $d^2\mathrm{K}(0)(v, v)$, as far as the latter is positive. But $d^2\mathrm{K}(0)(v, v)$ is bounded from above for any $v \in \mathbf{S}^1$ and uniformly bounded from below by a positive constant. By invariance under translations, $p_\mathrm{Z}$ is independent on $x$ as well as $\mathbf{E}(|\det \mathrm{Z}'(x)| | \mathrm{Z}(x) = 0)$, so that

$$\exists \mathrm{C} > 0,\ \forall v \in \mathbf{S}^1, \quad \mathbf{E}\big(\#\mathrm{C}_\varepsilon(\theta, s, v, f)\big) \leqslant \mathrm{C}\,\mathrm{Vol}(\mathrm{B}),$$

hence the result. □

We can now give the prove of Theorem 1.6.

*Proof of Theorem 1.6.* — Let $N \in \mathbf{N}^*$ be the number of directions of the edges of the lattice $\mathcal{T}$, and choose $v \in \mathbf{S}^1$ one of these directions. Fix $\eta > 0$ and for any $s \geqslant 1$, $\delta \in ]0, 1[$, define the functions

$$\lambda(s) = \mu\left(\frac{\eta}{6}\right)\left(\frac{\delta}{3N}\right)^{\kappa(\eta/6)}(2s)^{-2-\frac{\eta}{6}} \quad \text{and} \quad k(s) = \frac{C_1}{(\frac{\delta}{3N})}\sqrt{\log(2s)},$$

where $C_1$, $\kappa$ and $\mu$ are given by Lemma 5.2 and Lemma 5.3 respectively. Let us consider the event

$$\Omega_\delta = \Big\{\min_{x \in B_{2s}} \max\big(|f(x)|, |df(x)|\big) \geqslant \lambda(s)\Big\} \cap \Big\{\|f\|_{C^2(B_{2s})} \leqslant k(s)\Big\}. \tag{5.4}$$

By Lemmas 5.2 and 5.3 and Markov inequality, we have

$$\forall s \geqslant 1, \quad \mathbf{P}[\Omega_\delta] \geqslant 1 - \frac{2\delta}{3N}.$$

Let $\psi$ be the function defined by

$$\forall s \geqslant 1, \quad \psi(s) = (k/\lambda)(s) \in \mathbf{R}_+^*.$$

Note that there exists $c(\eta) > 0$, such that

$$\psi = c(\eta)\delta^{-1-\kappa(\eta/6)} s^{2+\eta/6}\sqrt{\log(2s)}. \tag{5.5}$$

Choose $s_1(\eta) \geqslant 2$, such that for any $\delta \in ]0, 1[$ and any $s \geqslant s_1(\eta)$, $\psi(s) \geqslant 1$, and define for every $s \geqslant 1$,

$$\varepsilon_1(s) = \left(\frac{1}{4}\psi^{-1}(s)\right)^2.$$

Note that if $\mu_{\mathcal{T}} > 0$ denotes the length of the largest edge in $\mathcal{E}$, then

$$\forall s \geqslant s_1(\delta, \eta), \ \forall \varepsilon \leqslant \varepsilon_1(s)/\mu_{\mathcal{T}}, \ \forall e \in \varepsilon\mathcal{E}, \ \forall x \in e, \quad e \subset b\big[x, \varepsilon_1(s)\big] \subset B_{2s},$$

where for any $(x_1, x_2) \in \mathbf{R}^2$ and any $\varepsilon > 0$,

$$b\big[(x_1, x_2), \varepsilon\big] = \big\{(y_1, y_2) \in \mathbf{R}^2, \max(|y_1 - x_1|, |y_2 - x_2|) \leqslant \varepsilon\big\},$$

see (A.2) below. Let us rotate the axes $(Ox)$ and $(Oy)$ such that the direction $v$ is horizontal. Fix $s \geqslant s_1(\delta, \eta)$, $\varepsilon \leqslant \varepsilon_1(s)/\mu_{\mathcal{T}}$, an horizontal edge $e \in \varepsilon\mathcal{E}$ intersecting $B_s$ and $x \in e \cap f^{-1}(0)$. Since

$$\max\left(\left|\frac{\partial f}{\partial x_1}(x)\right|, \left|\frac{\partial f}{\partial x_2}(x)\right|\right) \geqslant \frac{|df(x)|}{\sqrt{2}},$$

under the event $\Omega_\delta$ defined by (5.4), we can apply the quantitative version of the implicit function theorem given by Corollary A.3. It states that the nodal subset $f^{-1}(0) \cap b[x, \varepsilon_1(s)] \subset B_{2s}$ is the restriction of a graph over $(Oy)$ or $(Ox)$. In the first case, $f^{-1}(0) \cap b$ crosses $e$ at most once, that is $\#(f^{-1}(0) \cap e) \leqslant 1$.

Let us now consider the second case. Suppose that $f^{-1}(0)$ crosses at least two times the horizontal edge $e$, which we can assume to be part of $(Ox)$, and denote by $\phi$ the implicit function over $b[x, \varepsilon_1(s)] \cap (Ox) \supset e$, whose graph equals $f^{-1}(0) \cap b[x, \varepsilon_1(s)]$. Then, by Rolle's theorem, there exists a critical point $y \in e$ of $\phi_{|e}$ and by Taylor's theorem applied to $\phi_{|e}$ between $y$ and $x$, we obtain

$$|\phi(y)| \leqslant \frac{\mu_{\mathcal{T}}^2 \varepsilon^2}{2} \|\phi''\|_{L^\infty(e)} \leqslant 50 \mu_{\mathcal{T}}^2 \varepsilon^2 \psi^3(s),$$

where we used estimate (A.3) of Corollary A.3. This implies that $y \in C_\varepsilon(\theta(\varepsilon, s), 2s, v, f)$, see (5.3) in Lemma 5.5 for the definition of $C_\varepsilon$, with $\theta$ defined by

$$\forall \varepsilon > 0,\ s \geqslant 1, \quad \theta(\varepsilon, s) = 50 \mu_\tau^2 \varepsilon^2 \psi^3(s). \tag{5.6}$$

Now, denote by $\Omega'_\delta$ the event

$$\Omega'_\delta = \left\{ C_\varepsilon\big(\theta(\varepsilon, s), 2s, v, f\big) = \emptyset \right\},$$

and for every $s \geqslant 1$, let

$$\varepsilon_2(s) = \left(\frac{\delta}{3N}\right)\left(50 \mu_\tau^2 \beta s^2 \psi^3(s)\right)^{-1},$$

where $\beta$ is given by Lemma 5.5. By the latter, for any $s \geqslant 1$,

$$\forall \varepsilon \leqslant \varepsilon_2(s), \quad \mathbf{P}\big[\Omega'_\delta\big] \geqslant 1 - \frac{\delta}{3N}.$$

In conclusion, for any $s \geqslant s_1(\eta)$ and any $\varepsilon \leqslant \min(\varepsilon_1(s)/\mu_{\mathcal{T}}, \varepsilon_2(s))$, under the event $\Omega_\delta \cap \Omega'_\delta$ which has probability greater than $1 - \delta/N$, the nodal subset $f^{-1}(0) \cap B_s$ crosses at most once any edge $e \in \varepsilon\mathcal{V}$ in the direction $v$. After consideration of the N directions, this happens for any edge in $\varepsilon\mathcal{E} \cap B_s$ with probability at least $1 - \delta$. Lastly, by (5.5), there exists $C(\eta) > 0$,

$$\forall s \geqslant s(\eta), \quad \min\big(\varepsilon_1(s)/\mu_{\mathcal{T}}, \varepsilon_2(s)\big) \geqslant C(\eta) \delta^{4+3\kappa(\eta/6)} s^{-8-\eta},$$

which is the result. $\square$

**5.3.** *Proof of Theorem 1.4.* — We will prove the more general following theorem:

*Theorem* **5.7.** — *There exists $\alpha_0 > 0$, such that the following holds. For any random Gaussian field $f : \mathbf{R}^2 \to \mathbf{R}$ satisfying conditions (3.1), (3.2), (3.3), (3.4) for $\alpha \geqslant \alpha_0$ and (3.5), there exists $\gamma > 0$ and $\mathrm{C} > 0$ such that for any $1 \leqslant s < t$,*

$$\pi(s,t) \leqslant \mathrm{C}(s/t)^{\gamma}.$$

*Proof.* — Fix $1 \leqslant s < t$, and let $\mathcal{T}$ be a periodic symmetric triangulation. By Theorem 1.6, for any $\eta > 0$, there exist $\alpha(\eta)$, $s(\eta)$, $\mathrm{C}(\eta)$, such that for any $\delta > 0$, if

$$\varepsilon = \mathrm{C}(\eta)\delta^{\alpha(\eta)}\Big(\max\big(t, s(\eta)\big)\Big)^{-8-\eta}, \tag{5.7}$$

we have

$$\pi(s,t) \leqslant \pi_\varepsilon(s,t) + \delta, \tag{5.8}$$

where $\pi_\varepsilon(s,t)$ denotes the probability that there is a one-arm event for the $\varepsilon$-discretized model. Now we follow the proof of Lemma 2.10. By Proposition 4.1, there exists constants $\mathrm{C}, \mathrm{C}' > 0$ depending only on the parameters of the correlation function

$$\begin{aligned}
\pi_\varepsilon(s,t) &\leqslant \mathbf{P}\Big[\bigcap_{i\in\mathbf{N}, s\leqslant 5^i\sqrt{st}\leqslant t/2} \mathcal{A}^c_{5^i\sqrt{st}}\Big] \\
&\leqslant \mathbf{P}[\mathcal{A}^c_{\sqrt{st}}]\mathbf{P}\Big[\bigcap_{1\leqslant i\leqslant \lfloor \log_5(\frac12\sqrt{\frac{t}{s}})\rfloor} \mathcal{A}^c_{5^i\sqrt{st}}\Big] + \mathrm{C}\varepsilon^{-16/5}t^{16/5}(\sqrt{st})^{-\alpha/5} \\
&\leqslant \prod_{0\leqslant i\leqslant \lfloor \log_5(\frac12\sqrt{\frac{t}{s}})\rfloor} \mathbf{P}[\mathcal{A}^c_{5^i\sqrt{st}}] + \mathrm{C}'\varepsilon^{-16/5}t^{16/5}(\sqrt{st})^{-\alpha/5}\log t.
\end{aligned}$$

By Remark 4.10, using the symmetry of the field and again Theorem 1.6, this implies that there exists $c > 0$ and $\mathrm{C}'' > 0$, such that

$$\pi_\varepsilon(s,t) \leqslant \mathrm{C}''(1-c+\delta)^{\log\sqrt{t/s}} + \mathrm{C}''\varepsilon^{-16/5}t^{16/5}(\sqrt{st})^{-\alpha/5}\log t \tag{5.9}$$

Fix $\eta = 1$ and choose $\delta = \min(c/2, s/t)$. Combining (5.7), (5.8) and (5.9), we obtain the result. □

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## Appendix A

### A.1 Abstract Wiener spaces

Our main field of interest is the stationary Gaussian field with Gaussian covariance function. While it makes sense in itself, there is a natural way to view it as "the Gaussian in the Bargmann-Fock space", even though this is a priori not well-defined in infinite dimension. This appendix follows [27].

*The definition.* — Let $(\mathcal{H}, \|\cdot\|)$ be a Hilbert separable space of infinite dimension. Note the scalar product induces a natural Gaussian measure $\mu_G$ on any finite dimensional subspace $G \subset \mathcal{H}$. Let $N : \mathcal{H} \to \mathbf{R}^+$ be another norm, which satisfies the following condition: for any $\varepsilon > 0$, there exists a finite dimension subspace $H_\varepsilon \subset \mathcal{H}$, such that for any finite dimensional subspace $H_1$ of $H_\varepsilon^\perp$,

$$\mu_{H_1}\Big(\big\{x \in H_1, N(x) > \varepsilon\big\}\Big) < \varepsilon. \tag{A.1}$$

Such a norm is called *measurable* in the literature. Denote by B the completion of $\mathcal{H}$ for N: then there is a unique measure $\mu$ on B than agrees with $\mu_G$ on any cylindrical event based on G. More precisely: for every finite-dimensional subspace $G \subset \mathcal{H}$, and for every measurable $A \subset G$, letting $\tilde{A}$ denote the closure in B of $A + G^\perp$ in B, one has $\mu(\tilde{A}) = \mu_G(A)$.

The measured Banach space $(B, N, \mu)$ is called the *abstract Wiener space* associate to $\mathcal{H}$, relative to the measurable norm N. Of course if $\mathcal{H}$ is finite-dimensional, then $B = \mathcal{H}$, $N = \|\cdot\|$ and $\mu = \mu_{\mathcal{H}}$ is the standard Gaussian measure on $\mathcal{H}$. If on the other hand $\mathcal{H}$ is infinite-dimensional, then it is negligible for the Wiener measure, *i.e.* $\mu(\mathcal{H}) = 0$.

*The canonical example.* — If $(\mathcal{H}, \|\cdot\|_\nabla) = H^1([0, 1])$ equipped with the Dirichlet inner product defined by

$$\|f\|_\nabla^2 := \int_0^1 |f'(x)|^2 dx$$

and if the measurable norm is chosen to be the supremum norm, $N(f) = \|f\|_{L^\infty([0,1])}$ (which is indeed measurable), then $B = C(0, 1)$ and $\mu$ is the law of standard Brownian motion. In other words, the abstract Wiener space in this setup is the usual Wiener space of stochastic analysis.

*The Bargmann-Fock Wiener space.* — For any $R > 0$, we denote by $D(0,R) \subset \mathbf{R}^2$ the centered disc of radius R and $N_R$ the semi-norm on the real Bargmann-Fock space $\mathcal{F}$ given by $N_R = \|\cdot\|_{L^\infty(B(0,R))}$. This in fact defines a norm on $\mathcal{F}$, because by analytic continuation, if $N_R(f) = 0$, then $f$ is identically 0 in $\mathbf{R}^2$.

*Lemma* **A.1.** — *For any* $R > 0$, *the norm* $N_R$ *satisfies condition* (A.1), i.e. *it is measurable.*

*Proof.* — Fix $\varepsilon > 0$, $N \in \mathbf{N}$ and define $H_N = \mathbf{R}_N[x_1, x_2] \subset \mathcal{F}$ the subspace of real polynomials of degree less than N in two variables. Let $H_1 \subset H_N^\perp$ be any finite dimensional subspace orthogonal to $H_N$. Any $f \in H_1$ can be written as

$$\forall x = (x_1, x_2) \in \mathbf{R}^2, \quad f(x) = \sum_{i+j>N} a_{ij} \frac{x_1^i x_2^j}{\sqrt{i!j!}},$$

where $\sum a_{ij}^2 < \infty$ and the sum is locally uniformly convergent. By the triangle inequality and the Cauchy-Schwarz inequality applied to each homogeneous component of $f$:

$$\begin{aligned} N_R(f) &\leqslant \sum_{k>N} R^k \sum_{i+j=k} \frac{|a_{ij}|}{\sqrt{i!j!}} \\ &\leqslant \sum_{k>N} R^k \Big(\sum_{i+j=k} a_{ij}^2\Big)^{1/2} \Big(\sum_{i+j=k} \frac{1}{i!j!}\Big)^{1/2} = \sum_{k>N} \frac{2^{k/2} R^k}{\sqrt{k!}} \|f_k\|_{BF} \end{aligned}$$

where $f_k$ is the component of degree $k$ of $f$, namely

$$f_k = \sum_{i+j=k} a_{ij} \frac{x_1^i x_2^j}{\sqrt{i!j!}}.$$

By a simple union bound, we then get for $f \sim \mu_{H_1}$:

$$\begin{aligned} P\big[N_R(f) > \varepsilon\big] &\leqslant \sum_{k>N} P\Big[\frac{2^{k/2} R^k}{\sqrt{k!}} \|f_k\|_{BF} > \varepsilon 2^{N-k}\Big] \\ &\leqslant \sum_{k>N} P\Big[\|f_k\|_{BF}^2 > \frac{\varepsilon^2 4^{N-k} k!}{2^k R^{2k}}\Big] \end{aligned}$$

which leads to the upper bound

$$P\big[N_R(f) > \varepsilon\big] \leqslant \sum_{k>N} \frac{2^k R^{2k}}{\varepsilon^2 4^{N-k} k!} E\big[\|f_k\|_{BF}^2\big].$$

Since $f_k$ is a Gaussian element of a vector space of dimension $k+1$ we finally obtain

$$P\big[N_R(f) > \varepsilon\big] \leqslant \sum_{k>N} \frac{(k+1)2^k R^{2k}}{\varepsilon^2 4^{N-k} k!} \leqslant \frac{1}{\varepsilon^2 4^N} \sum_{k>N} \frac{16^k R^{2k}}{k!} \leqslant \frac{e^{16R^2}}{\varepsilon^2 4^N}.$$

Choosing N large enough can make the bound arbitrarily small, which is what we had to prove. □

We can construct a family of abstract Wiener spaces $(B_R, N_R)$ associated to these norms. In fact we get something much better: by regularity of analytic continuation, the norms $N_R$ are pairwise equivalent, and the completion we obtain therefore does not actually depend on R. The natural extension of the notion of abstract Wiener space with which we work is therefore the space $L^\infty_{loc}$ equipped with the collection of seminorms $(N_R)$ and with the measure $\mu$ obtained from Gross's construction (which does not depend on R).

## A.2 A quantitative implicit function theorem

For each $\delta > 0$ and any $x = (x_1, x_2) \in \mathbf{R}^p \times \mathbf{R}^q$, define

$$b[x,\delta] = \big\{(y_1, y_2) \in \mathbf{R}^p \times \mathbf{R}^q,\ \|y_1 - x_1\| \leqslant \delta,\ \|y_2 - y_1\| \leqslant \delta\big\}. \tag{A.2}$$

We will use the following quantitative refinement of the usual implicit function theorem, which follows from a careful bookkeeping of the constants in the standard fixed point argument.

*Theorem* **A.2.** — *Let $f : \mathbf{R}^{p+q} \to \mathbf{R}^q$ be a $C^2$ function and $x = (x_1, x_2) \in \mathbf{R}^p \times \mathbf{R}^q$ such that $f(x) = 0$ and the partial derivative $d_{x_2} f(x) : \mathbf{R}^q \to \mathbf{R}^q$ is invertible. Choose $\delta > 0$ such that*

$$\sup_{y\in b[x,\delta]} \left\| \mathrm{Id}_{\mathbf{R}^q} - \big(d_{x_2} f(x)\big)^{-1} d_{x_2} f(y) \right\| \leqslant 1/2.$$

*Let* $C = \sup_{y\in b[x,\delta]} \|d_{x_1} f(y)\|$, $M = \|d_{x_2} f(x)^{-1}\|$,

$$\delta' = \delta(2MC)^{-1}$$

*and* $I_{\delta'} := \{y_1 \in \mathbf{R}^p : \|y_1 - x_1\| < \delta'\}$. *Then there exists* $\phi : I_{\delta'} \to \mathbf{R}^q$ *a* $C^2$ *function such that*

$$\forall y = (y_1, y_2) \in \mathbf{R}^p \times \mathbf{R}^q, \quad \|y_1 - x_1\| < \delta', \quad \|y_2 - x_2\| < \delta,$$
$$f(y) = 0 \Leftrightarrow y_2 = \phi(y_1).$$

*Corollary* **A.3.** — *Let $k>0$, $\lambda>0$ be such that $k/\lambda \geqslant 1$, $\mathrm{U}\subset \mathbf{R}^2$ an open set, $f$ be a $\mathrm{C}^2$ function $f:\mathbf{R}^2\to\mathbf{R}$ such that $\|df\|_{\mathrm{C}^1(\mathrm{U})}\leqslant k$. Fix $x\in\mathbf{R}^2$ satisfying $f(x)=0$ and $|\partial_{x_2}f(x)|\geqslant \frac{\lambda}{\sqrt{2}}$. Then if*

$$\varepsilon=\left(\frac{\lambda}{4k}\right)^2,$$

*the nodal set $b[x,\varepsilon]\cap f^{-1}(0)$ is the graph over $\mathrm{I}_\varepsilon$ of a $\mathrm{C}^2$ function $\phi$. Moreover,*

$$\|\phi''\|_{\mathrm{L}^\infty(\mathrm{I}_\varepsilon)}\leqslant 100\left(\frac{k}{\lambda}\right)^3. \tag{\textbf{A.3}}$$

*Proof of Corollary A.3.* — Fix $\delta=\frac{\lambda}{4\sqrt{2}k}$. By the mean value theorem,

$$\forall y\in b[x,\delta],\quad |\partial_{x_2}f(y)-\partial_{x_2}f(x)|\leqslant\sqrt{2}\delta k=\frac{\lambda}{4}, \tag{\textbf{A.4}}$$

so that

$$\sup_{y\in b[x,\delta]}\left|1-\left(\partial_{x_2}f(x)\right)^{-1}\partial_{x_2}f(y)\right|\leqslant 1/(2\sqrt{2})\leqslant 1/2.$$

Using the notations of Theorem A.2, we thus have $\mathrm{M}\leqslant\frac{\sqrt{2}}{\lambda}$, $\mathrm{C}\leqslant k$, so that $\delta'\geqslant(\frac{\lambda}{4k})^2$. Since $k\geqslant\lambda$, $\delta\geqslant(\frac{\lambda}{4k})^2$, which implies the first assertion. Now, since

$$\forall t\in\mathrm{I}_{\delta'},\quad f\big(t,\phi(t)\big)=0,$$

then $\forall t\in\mathrm{I}_{\delta'}$, $\partial_{x_1}f+\partial_{x_2}f\phi'(t)=0$ so that $|\phi'(t)|\leqslant\frac{k}{\lambda/\sqrt{2}-\lambda/4}\leqslant\frac{4k}{\lambda}$ by (A.4) and the hypothesis on $\partial_{x_2}f(x)$, and

$$\forall t\in\mathrm{I}_{\delta'},\quad \partial^2_{x_1^2}f+2\partial^2_{x_1x_2}f\phi'(t)+\partial^2_{x_2^2}f\phi'(t)^2+\partial_{x_2}f\phi''(t)=0.$$

This implies $|\phi''(t)|\leqslant\frac{4k}{\lambda}(1+\frac{8k}{\lambda}+\frac{16k^2}{\lambda^2})\leqslant 100(k/\lambda)^3$ since $k\geqslant\lambda$. □

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