ON THE HYPERBOLICITY OF GENERAL HYPERSURFACES

by Damian BROTBEK

ABSTRACT

In 1970, Kobayashi conjectured that general hypersurfaces of sufficiently large degree in $\mathbf{P}^n$ are hyperbolic. In this paper we prove that a general sufficiently ample hypersurface in a smooth projective variety is hyperbolic. To prove this statement, we construct hypersurfaces satisfying a property which is Zariski open and which implies hyperbolicity. These hypersurfaces are chosen such that the geometry of their higher order jet spaces can be related to the geometry of a universal family of complete intersections. To do so, we introduce a Wronskian construction which associates a (twisted) jet differential to every finite family of global sections of a line bundle.

1. Introduction

A smooth projective variety X over the field of complex numbers is said to be *Brody hyperbolic* if there is no non-constant holomorphic map $f : \mathbf{C} \to X$. In view of a result of Brody [3], in our situation (when X is compact), this is equivalent to saying that X is *Kobayashi hyperbolic*, and we will simply use the word *hyperbolic* in what follows. In [34, 35], Kobayashi conjectured: *a general hypersurface in* $\mathbf{P}^n$ *of sufficiently large degree is hyperbolic*. When $n = 2$, this conjecture follows from the fact that a curve is hyperbolic if and only if its genus is greater or equal to two, which for a smooth curve in $\mathbf{P}^2$ is the case if and only if its degree is greater or equal to four.

Before considering the general situation, one might wonder if there exist examples of hyperbolic hypersurfaces in $\mathbf{P}^n$ with $n \geqslant 3$. The first such examples in $\mathbf{P}^3$ were constructed by Brody and Green [4] as hypersurfaces defined by equations of the form

$$X_0^{2r} + X_1^{2r} + X_2^{2r} + X_3^{2r} + aX_0^r X_1^r + bX_0^r X_2^r = 0,$$

with $r \geqslant 25$ and general $a, b \in \mathbf{C}$. Afterwards, many authors have provided examples of this nature, see for instance [19, 30, 44, 56] and the work of Masuda and Noguchi [39], where a considerable amount of examples in any dimension is given. See [66] for more details. It is known that hyperbolicity is an open property with respect to the Euclidean topology (see [3]). It is however unknown if it is open with respect to the Zariski topology. Therefore, one cannot deduce the hyperbolicity of general hypersurfaces of degree d from the existence of a hyperbolic hypersurface of degree d.

For the case $n = 3$, the first proofs of the Kobayashi conjecture for very general hypersurfaces were provided in [41] (for hypersurfaces of degree $d \geqslant 36$) and [20] (for $d \geqslant 21$), and relied among other things on the ideas of McQuillan [40] about the entire leaves of foliations on surfaces. The bound was later improved to $d \geqslant 18$ in [51].

In a more algebraic direction, one can study the positivity of the canonical bundle of subvarieties of general hypersurfaces. Recall that the Green-Griffiths-Lang conjecture [32, 38] predicts that varieties of general type are weakly hyperbolic (where we say that

DOI 10.1007/s10240-017-0090-3

a variety X is *weakly hyperbolic* if all its entire curves lie in a subvariety $Z \subsetneq X$). A positive answer to this conjecture would in particular imply that a smooth projective variety of general type is hyperbolic if all of its subvarieties are of general type. The fact that all subvarieties of very general hypersurfaces of large degree in $\mathbf{P}^n$ are of general type was established by the work of Clemens [8], Ein [28, 29] and Voisin [62], later improved by Pacienza [50].

In [59], Siu generalized Voisin's variational method from [62] to higher order jet spaces, and outlined a strategy to prove Kobayashi's conjecture. This motivated a lot of research over the last decade [9, 10, 23–25, 27, 42, 51, 54], which culminated with the work of Diverio, Merker and Rousseau [25], and the proof of the weak hyperbolicity of general hypersurfaces in $\mathbf{P}^n$ of degree $d \geqslant 2^{(n-1)^5}$. Building on [25], Diverio and Trapani [27] proved that the Kobayashi conjecture holds for very general hypersurfaces in $\mathbf{P}^4$ of degree $d \geqslant 593$. The bound of the theorem of [25] was later improved by different authors [2, 16], the current best bound being $d \geqslant (5n)^2 n^n$ [9]. We refer to [52] for more details on this approach. More recently, in [60], Siu provided more details to the strategy outlined in [59] in order to complete his proof of the Kobayashi conjecture.

Lastly, Demailly developed another approach towards Kobayashi's conjecture [12, 18] based on his work on the Green-Griffiths-Lang conjecture [16, 17].

In view of the work of Zaidenberg [65], and on the aforementioned works [8, 28, 29, 50, 62], one can expect a possible bound in the Kobayashi conjecture to be $d \geqslant 2n-1$ for $n \geqslant 3$.

The goal of the present paper is to provide an alternative approach to the Kobayashi conjecture in order to prove the following statement.

Main Theorem. — Let X *be a smooth projective variety. For any ample line bundle* A *on* X, *there exists* $d_0 \in \mathbf{N}$ *such that for any* $d \geqslant d_0$, *a general hypersurface* $H \in |A^d|$ *is hyperbolic.*

Note that when $X = \mathbf{P}^n$ and $A = \mathscr{O}_{\mathbf{P}^n}(1)$, this is precisely the Kobayashi conjecture. On the other hand, the Kobayashi conjecture implies our main result for degree d sufficiently large and sufficiently divisible. Indeed, for d sufficiently large, A^d induces an embedding $X \hookrightarrow \mathbf{P}^N$ such that $A^d = \mathscr{O}_{\mathbf{P}^N}(1)|_X$. Then, taking d' sufficiently large, the statement for elements in $|A^{dd'}|$ is reduced to the statement for elements in $|\mathscr{O}_{\mathbf{P}^N}(d')|$. Let us emphasize that our result holds for general hypersurfaces and not only very general ones.

The proof we present here is not effective on d_0 because of two noetherianity arguments: the first concerns the value of the integer $m_\infty(P_k V, L)$ in formula (9), the second concerns the value of the integer m_J in formula (40). However, shortly after a first version of the present paper was made available on arXiv, Ya Deng [21] (see also [22]) was able to render both arguments effective, and obtained the bound $d_0 = n^{n+1}(n+1)^{2n+5}$ when A is very ample (where $n = \dim X$).

The main tool of our proof is the use of jet differential equations. Those can be seen as higher order analogues of symmetric differential forms and provide obstructions to the existence of entire curves [14, 15, 32, 61]. A fruitful way to produce jet differential equations on a given variety is to use the Riemann-Roch theorem (see for instance [32, 53]) or Demailly's holomorphic Morse inequalities [13] (see for instance [23, 24], and also [16, 43]). However, our proof relies on another construction, described below.

A general strategy towards proving a hyperbolicity statement is to construct jet differential equations on the variety under consideration and then to control their base locus in an adequate jet space. This strategy has already been carried out successfully for instance in [25] and [60].

Considering jets of order one, recall that a conjecture of Debarre [11] predicts that *a general complete intersection in* $\mathbf{P}^n$ *of high multidegree and of codimension larger than its dimension, has ample cotangent bundle*. A natural way to approach this conjecture is to construct symmetric differential forms (jet differential equations of order one) on the complete intersection under consideration, and to control their base locus. This rises a connection between the Kobayashi and the Debarre conjecture which motivated a conjecture of Diverio and Trapani [27]. This connection was investigated in [5], where among other things, we used the strategy of [59] and the ideas of [25] to prove the conjecture of Debarre for complete intersection surfaces. Later, in [6], we proved a higher dimensional result towards this conjecture. To do so, we used the openness property of ampleness to reduce the statement for general complete intersections of a given multi-degree to the construction of an example. This example was constructed by intersecting (many) particular deformations of Fermat type hypersurfaces, on which we were able to produce explicit symmetric differential forms. Afterwards, in [63] (see also [64]), Xie was able to prove the Debarre conjecture (with an explicit bound on the degree) by, among other things, generalizing the symmetric differential forms constructed in [6] to a wider class of complete intersections. Independently, in a joint work with Darondeau [7], we gave another proof of the Debarre conjecture (with no explicit bound on the degree and with a restriction on the possible multidegrees), by providing a geometric interpretation of the cohomological computations of [6].

In the present paper, we generalize the approach developed in [7] to higher order jet spaces. To simplify the exposition, we will temporarily restrict ourselves to the case $X = \mathbf{P}^n$ and $A = \mathscr{O}_{\mathbf{P}^n}(1)$.

As mentioned above, it is currently unknown if hyperbolicity is an open property in the Zariski topology. In order to prove the main theorem, we thus construct an example of a hypersurface satisfying a certain ampleness property (∗), which implies hyperbolicity and which is a Zariski open property. The statement for general hypersurfaces will then follow from this particular example.

To construct this example, we use hypersurfaces of the same type as the ones used in [7]. Consider degree d homogenous polynomials in $\mathbf{C}[z_0, \dots, z_n]$ of the form

$$F(\mathbf{a}) = \sum_{\substack{I=(i_0,\dots,i_n)\\ i_0+\cdots+i_n=\delta}} a_I z^{(r+k)I} \tag{1}$$

where we use the multi-index notation $z^{(r+k)I} := z_0^{(r+k)i_0}\cdots z_n^{(r+k)i_n}$ and where the a_I are homogenous polynomials of degree ε, so that $d = \varepsilon + (r+k)\delta$.

Let us motivate this choice of equations by giving a rough idea about how we use the special form of the polynomials $F(\mathbf{a})$ to construct jet differential equations on the associated hypersurfaces. As we will see, the higher order differentials of $F(\mathbf{a})$, can be written, locally, as

$$
(2)\qquad \begin{cases} F(\mathbf{a}) = \sum_I a_I z^{(r+k)I} = \sum_I \alpha_I^0 z^{rI} = \sum_I \alpha_I^0 T^I \\ d^{[1]}F(\mathbf{a}) = \sum_I \tilde{a}_I^1 z^{(r+k-1)I} = \sum_I \alpha_I^1 z^{rI} = \sum_I \alpha_I^1 T^I \\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ d^{[k]}F(\mathbf{a}) = \sum_I \tilde{a}_I^k z^{(r+k-k)I} = \sum_I \alpha_I^k z^{rI} = \sum_I \alpha_I^k T^I. \end{cases}
$$

Here, $\alpha_I^p = \tilde{a}_I^p z^{(r+k-p)I}$, $\alpha_I^0 = a_I z^{kI}$ and $T_i = z_i^r$ for any $0 \leqslant i \leqslant n$, where the $\tilde{a}_I^p$ should be thought of as differential forms of order p. One should think of the equations on the left hand side as the equations defining a suitable kth order jet space $H_{\mathbf{a},k}$ of the hypersurface $H_{\mathbf{a}} \subset \mathbf{P}^n$ defined by $F_{\mathbf{a}}$. Considering $[T_0, \ldots, T_n]$ as homogenous coordinates on $\mathbf{P}^n$, one should think of the equations on the right hand side as the equations of the universal family $\mathscr{Y} \subset \mathrm{Gr}_{k+1}(H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(\delta))) \times \mathbf{P}^n$ of complete intersections of codimension $k+1$ and multidegree $(\delta, \ldots, \delta)$ in $\mathbf{P}^n$. Here Gr_{k+1} denotes the Grassmannian of $(k+1)$-dimensional subspaces. The key point is that a suitable interpretation of (2) implies that every element of $H^0(\mathscr{Y}, q_1^*\mathscr{Q}^m \otimes q_2^*\mathscr{O}_{\mathbf{P}^n}(-1))$ induces a jet differential equation on $H_{\mathbf{a}}$ (where $\mathscr{Q}$ denotes the Plücker line bundle on the Grassmannian and q_1, q_2 denote the canonical projections). But when $k+1 \geqslant n$, the morphism q_1 is generically finite, so that $q_1^*\mathscr{Q}$ is big and nef. Therefore, for large m, $H^0(\mathscr{Y}, q_1^*\mathscr{Q}^m \otimes q_2^*\mathscr{O}_{\mathbf{P}^n}(-1))$ contains many elements, from which we infer the existence of many jet differential equations on $H_{\mathbf{a}}$.

The outline of the paper is the following. Section 2 is devoted to a Wronskian construction which is one of the main tools of this paper. First, the needed properties concerning the Demailly-Semple jet tower are recalled, then the Wronskian associated to families of global sections of a line bundle is defined. This allows us to introduce an ideal sheaf on each stage of the jet tower, whose blow-up satisfies several functorial properties. The aforementioned property (∗), which says that a certain line bundle on such a blow-up is ample, is then introduced. Section 3 is devoted to the proof of our main result. After the hypersurfaces $H_{\mathbf{a}}$ are introduced, the above relationship between the jet space of $H_{\mathbf{a}}$ and the universal family $\mathscr{Y}$ is formalized. This is the main technical part of our paper. Once this is done, we explain how the geometry of $\mathscr{Y}$ is used to prove that the hypersurface $H_{\mathbf{a}}$ satisfies (∗) and therefore conclude the proof of our main result.

In this article, we will work over the field of complex numbers $\mathbf{C}$. While the objects we consider are mostly of algebraic nature, we work in the analytic category because this is needed on a few occasions. Given a vector bundle E on a variety X, the projectivization of *lines* in E is denoted by P(E). The tangent bundle of a smooth variety X is denoted by T_X and its cotangent bundle by T_X^*. A property is said to hold for a general member of an algebraic family of projective varieties $\mathscr{X} \to \mathrm{T}$ if it holds for each fiber over a non-empty *Zariski* open subset of T.

2. Wronskians on the Demailly-Semple jet tower

In this section we construct the main tool used in the proof of our main result, namely a suitable type of Wronskians on the Demailly-Semple tower. Wronskians provide a fundamental tool in the study of entire curves and in particular in Nevanlinna theory (see for instance [49]). A fruitful way to construct Wronskians is to use a connection satisfying some regularity assumption as for instance in [19, 30, 44, 48, 58]. By contrast, the Wronskians we introduce in this paper, are associated to sections of a given line bundle. This approach is certainly more classical, as it is mainly a reinterpretation of the Plücker coordinates of higher order osculating planes associated to projective curves [33]. In the one dimensional case, such objects where already studied for different purposes as for instance in [31, 37, 47].

2.1. *The Demailly-Semple jet tower.* — Let us first recall the results we need from Demailly's foundational work [14], in which the reader will find all the details of the results outlined here. Let X be an n-dimensional complex manifold and denote by $\mathrm{J}_k\mathrm{X} \xrightarrow{p_k} \mathrm{X}$ the kth order jet space of X. This is the set of equivalence classes of holomorphic maps $\gamma : (\mathbf{C}, 0) \to \mathrm{X}$, where $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1^{(p)}(0) = \gamma_2^{(p)}(0)$ for all $0 \leqslant p \leqslant k$ (the derivatives being computed in any coordinate chart). The class of γ in $\mathrm{J}_k\mathrm{X}$ is denoted by $[\gamma]_k$. The map p_k is defined by $p_k([\gamma]_k) := \gamma(0)$. The space $\mathrm{J}_k\mathrm{X}$ naturally possesses the structure of a $\mathbf{C}^{nk}$-fiber bundle. Indeed, coordinates $(z_1, \dots, z_n)$ on a chart $\mathrm{U} \subset \mathrm{X}$ induce coordinates

$$\left(z_1, \dots, z_n, z_1', \dots, z_n', \dots, z_1^{(k)}, \dots, z_n^{(k)}\right)$$

on $p_k^{-1}(\mathrm{U})$, where by definition, a jet $[\gamma]_k \in p_k^{-1}(\mathrm{U})$ has coordinates $(\gamma_1(0), \dots, \gamma_n(0), \dots, \gamma_1^{(k)}(0), \dots, \gamma_n^{(k)}(0))$.

A *directed manifold* is a pair (X, V) where X is a complex manifold and where $\mathrm{V} \subset \mathrm{T}_\mathrm{X}$ is a subbundle of T_X. In the present paper, we only need two special cases of this general framework: the *absolute case*, when we consider the directed variety $(\mathrm{X}, \mathrm{T}_\mathrm{X})$; the *relative case*, when we consider the directed variety $(\mathscr{X}, \mathrm{T}_{\mathscr{X}/\mathrm{T}})$, where $\mathscr{X} \to \mathrm{T}$ is a smooth projective morphism of quasi-projective varieties. But for the clarity of the exposition, we work in the generality of [14].

On a directed manifold (X, V), one defines $\mathrm{J}_k\mathrm{V} \xrightarrow{p_k} \mathrm{X}$ to be the subset $\mathrm{J}_k\mathrm{V} \subset \mathrm{J}_k\mathrm{X}$ of all k-jets of curves $\gamma : (\mathbf{C}, 0) \to \mathrm{X}$ tangent to V (*i.e.* $\gamma'(t) \in \mathrm{V}_{\gamma(t)}$ for all t in a neighborhood of 0). It can be shown that $\mathrm{J}_k\mathrm{V}$ is a subbundle of $\mathrm{J}_k\mathrm{X}$.

One denotes by $\mathbf{G}_k$ the group of germs of k-jets of biholomorphisms of $(\mathbf{C}, 0)$, namely

$$\mathbf{G}_k := \left\{\varphi : t \mapsto a_1 t + a_2 t^2 + \cdots + a_k t^k \;\middle|\; a_1 \in \mathbf{C}^* \text{ and } a_j \in \mathbf{C} \text{ for } j \geqslant 2\right\},$$

where composition is taken modulo t^{k+1}. Given a directed manifold (X, V), the group $\mathbf{G}_k$ naturally acts on $J_k V$ by the (right) action $\varphi \cdot [\gamma]_k := [\gamma \circ \varphi]_k$. For any $k, m \geqslant 1$, one can construct a locally free sheaf $E_{k,m}V^*$, the sheaf of *invariant jet differential equations of order k and degree m*, satisfying for any open $U \subset X$

$$E_{k,m}V^*(U) = \Big\{Q \in \mathscr{O}\big(p_k^{-1}(U)\big) \;\Big|\; Q\big(\varphi \cdot [\gamma]_k\big) = \varphi'(0)^m Q\big([\gamma]_k\big)$$
$$\forall [\gamma]_k \in p_k^{-1}(U),\ \forall \varphi \in \mathbf{G}_k\Big\}.$$

In the spirit of [55], Demailly also constructs, for each $k \geqslant 1$, a manifold $P_k V$ of dimension $n + k(r-1)$ (where $r = \operatorname{rank} V$) equipped with a rank r vector bundle V_k satisfying $P_{k+1}V = P(V_k)$, $X_0 = X$ and $V_0 = V$. We will refer to the sequence

$$\cdots \longrightarrow P_k V \xrightarrow{\pi_k} P_{k-1}V \xrightarrow{\pi_{k-1}} \cdots \longrightarrow P_1 V \xrightarrow{\pi_1} X_0 = X$$

as the *Demailly-Semple jet tower of* (X, V). In the absolute case (X, T_X), we will simply write $X_k := P_k T_X$, and in the relative case $(\mathscr{X}, T_{\mathscr{X}/T})$, we will write $\mathscr{X}_k^{\mathrm{rel}} := P_k T_{\mathscr{X}/T}$.

For each $k \geqslant 1$, $P_k V$ comes with a tautological line bundle $\mathscr{O}_{P_k V}(1)$, and more generally, for any $a_1, \ldots, a_k \in \mathbf{Z}$, we set

$$\mathscr{O}_{P_k V}(a_k, \ldots, a_1) := \mathscr{O}_{P_k V}(a_k) \otimes \pi_{k-1,k}^* \mathscr{O}_{P_{k-1}V}(a_{k-1}) \otimes \cdots \otimes \pi_{1,k}^* \mathscr{O}_{P_1 V}(a_1),$$

where for any $0 \leqslant p \leqslant k$, one writes $\pi_{p,k} := \pi_{p+1} \circ \cdots \circ \pi_k$.

From [14] §5, any germ of curve $\gamma : (\mathbf{C}, 0) \to X$ tangent to V can be lifted to a germ $\gamma_{[k]} : (\mathbf{C}, 0) \to P_k V$. Moreover, if one denotes by $J_k^{\mathrm{reg}}V := \{[\gamma]_k \in J_k V \mid \gamma'(0) \neq 0\}$ the space of *regular k-jets* tangent to V, then there exists a morphism $J_k^{\mathrm{reg}}V \to P_k V$, sending $[\gamma]_k$ to $\gamma_{[k]}(0)$, whose image is an open subset $P_k V^{\mathrm{reg}} \subset P_k V$ which can be identified with the quotient $J_k^{\mathrm{reg}}V/\mathbf{G}_k$ (see Theorem 6.8 in [14]). Let us mention moreover that $P_k V^{\mathrm{sing}} := P_k V \setminus P_k V^{\mathrm{reg}}$ is a divisor in $P_k V$.

From [14] Theorem 6.8, for any $k, m \geqslant 0$ one has

$$E_{k,m}V^* = (\pi_{0,k})_* \mathscr{O}_{P_k V}(m). \tag{3}$$

This isomorphism is described as follows. From Corollary 5.12 in [14], for any $w_0 \in P_k V$, there exists an open neighborhood U_{w_0} of w_0 and a family of germs of curves $(\gamma_w)_{w \in U_{w_0}}$ tangent to V, depending holomorphically on w, such that

$$(\gamma_w)_{[k]}(0) = w \quad \text{and} \quad (\gamma_w)'_{[k-1]}(0) \neq 0, \quad \forall w \in U_{w_0}. \tag{4}$$

The image of a given $Q \in E_{k,m}V^*(U)$ under the isomorphism (3) is the section $\sigma \in \mathscr{O}_{P_k(V)}(m)(\pi_{0,k}^{-1}(U))$ defined by

$$\sigma(w) = Q\big([\gamma_w]_k\big)\big((\gamma_w)'_{[k-1]}(0)\big)^{-m}.$$

Every non-constant entire curve $f : \mathbf{C} \to X$ tangent to V can be lifted to an entire curve $f_{[k]} : \mathbf{C} \to P_k V$ satisfying $f_{[k]}(t) \in P_k V^{\mathrm{reg}}$ if $f'(t) \neq 0$, so that in particular, the image of $f_{[k]}$ is not entirely contained in $P_k V^{\mathrm{sing}}$. The following fundamental result shows that the existence of jet differential equations vanishing along some ample divisor provides obstructions to the existence of entire curves.

Theorem **2.1** *(Demailly, Green-Griffiths, Siu-Yeung). — Let* X *be a smooth projective variety and* V *a subbundle of* T_X. *For any non-constant entire curve* $f : \mathbf{C} \to X$ *tangent to* V, *any ample line bundle* A *on* X, *any* $a_1, \dots, a_k \in \mathbf{N}$ *and any* $\omega \in H^0(P_k V, \mathscr{O}_{P_k V}(a_k, \dots, a_1) \otimes \pi_{0,k}^* A^{-1})$, *we have*

$$f_{[k]}(\mathbf{C}) \subset (\omega = 0).$$

Let us mention that in the case $k = 1$, a similar result was obtained by Noguchi in [46].

2.2. *Wronskians.* — We now describe the Wronskian construction on which we rely in the rest of this paper. Take an n-dimensional complex manifold X and an integer $k \geqslant 0$. Let us start with a local construction. Let U be an open subset of X. For every $0 \leqslant p \leqslant k$, one can define a $\mathbf{C}$-linear map

$$d_U^{[p]} : \mathscr{O}(U) \to \mathscr{O}\big(p_k^{-1}(U)\big)$$

by $d_U^{[p]} f([\gamma]_k) = (f \circ \gamma)^{(p)}(0)$ for every $f \in \mathscr{O}(U)$ and $[\gamma]_k \in p_k^{-1}(U) \subset J_k X$. One easily verifies that $d_U^{[p]} f$ is holomorphic and well defined. Indeed, given a chart in U with coordinates $\underline{z} = (z_1, \dots, z_n)$, by considering the induced coordinates $(\underline{z}, \underline{z}', \dots, \underline{z}^{(k)})$, one can describe $d_U^{[p]} f$ inductively as follows:

$$(5) \qquad d_U^{[0]} f = f \quad \text{and} \quad d_U^{[p+1]} f\big(\underline{z}, \dots, \underline{z}^{(k)}\big) = \sum_{m=0}^{p} \sum_{i=1}^{n} \frac{\partial d_U^{[p]} f}{\partial z_i^{(m)}} z_i^{(m+1)} \quad \text{for all } 0 \leqslant p < k,$$

from which the holomorphicity follows at once. Observe also that this expression implies that $d_U^{[p]} f([\gamma]_k)$ only depends on the jets of f at $x := \gamma(0)$ of order less or equal p, by which we mean that it only depends on the class of f in $\mathscr{O}_{X,x}/\mathfrak{m}_{X,x}^{p+1}$, where $\mathfrak{m}_{X,x}$ denotes the maximal ideal of $\mathscr{O}_{X,x}$. This remark will be used in the proof of Lemma 2.4. The map $d_U^{[p]}$ also satisfies the following generalized Leibniz rule:

$$d_U^{[p]}(fg) = \sum_{i=0}^{p} \binom{p}{i} d_U^{[i]}(f) d_U^{[p-i]}(g).$$

Using this differentiation rule, one can construct the Wronskian of any $(k+1)$ holomorphic functions $f_0, \dots, f_k \in \mathscr{O}(\mathrm{U})$ by

$$
(\mathbf{6}) \qquad \mathrm{W}_{\mathrm{U}}(f_0, \dots, f_k) := \begin{vmatrix} d_{\mathrm{U}}^{[0]} f_0 & \cdots & d_{\mathrm{U}}^{[0]} f_k \\ \vdots & \ddots & \vdots \\ d_{\mathrm{U}}^{[k]} f_0 & \cdots & d_{\mathrm{U}}^{[k]} f_k \end{vmatrix} \in \mathscr{O}\big(p_k^{-1}(\mathrm{U})\big).
$$

This object will be most crucial to us. Let us start by proving that it is an invariant jet differential equation. To ease our notation, in the rest of this paper, for any $k \in \mathbf{N}$, we set

$$
k' := \frac{k(k+1)}{2} = 1 + 2 + \cdots + k.
$$

Proposition **2.2.** — *With respect to the above notation, for any* $f_0, \dots, f_k \in \mathscr{O}(\mathrm{U})$*, we have*

$$
\mathrm{W}_{\mathrm{U}}(f_0, \dots, f_k) \in \mathrm{E}_{k,k'}\mathrm{T}_{\mathrm{X}}^{*}(\mathrm{U}).
$$

Proof. — Recall Faà Di Bruno's formula for holomorphic functions h, g in one variable such that $h \circ g$ is defined:

$$
(\mathbf{7}) \qquad (h \circ g)^{(p)}(0) = \sum_{i=1}^{p} \mathrm{P}_{p,i}(g) \cdot h^{(i)}\big(g(0)\big),
$$

where $\mathrm{P}_{p,i}(g) := \mathrm{B}_{p,i}(g'(0), \dots, g^{(p-i+1)}(0))$ and $\mathrm{B}_{p,i}$ denotes a Bell polynomial. Here, we only need to know that $\mathrm{P}_{p,p}(g) = g'(0)^p$. Take $[\gamma]_k \in p_k^{-1}(\mathrm{U})$ and $\varphi \in \mathbf{G}_k$. For any $f \in \mathscr{O}(\mathrm{U})$ and any $1 \leqslant p \leqslant k$, one has

$$
\begin{aligned}
d_{\mathrm{U}}^{[p]} f\big([\gamma \circ \varphi]_k\big) = (f \circ \gamma \circ \varphi)^{(p)}(0) &= \sum_{i=1}^{p} \mathrm{P}_{p,i}(\varphi)(f \circ \gamma)^{(i)}(0) \\
&= \mathrm{P}_{p,p}(\varphi) d_{\mathrm{U}}^{[p]} f\big([\gamma]_k\big) + \sum_{i=1}^{p-1} \mathrm{P}_{p,i}(\varphi) d_{\mathrm{U}}^{[i]} f\big([\gamma]_k\big) \\
&= \varphi'(0)^p d_{\mathrm{U}}^{[p]} f\big([\gamma]_k\big) + \sum_{i=1}^{p-1} \mathrm{P}_{p,i}(\varphi) d_{\mathrm{U}}^{[i]} f\big([\gamma]_k\big).
\end{aligned}
$$

Applying this formula to $f = f_0, \dots, f_k$ and performing elementary operations on the lines in (6) one obtains

$$
\mathrm{W}_{\mathrm{U}}(f_0, \dots, f_k)\big(\varphi \cdot [\gamma]_k\big) = \varphi'(0)^{1+2+\cdots+k} \mathrm{W}_{\mathrm{U}}(f_0, \dots, f_k)\big([\gamma]_k\big),
$$

and therefore, $\mathrm{W}_{\mathrm{U}}(f_0, \dots, f_k) \in \mathrm{E}_{k,k'}\mathrm{T}_{\mathrm{X}}^{*}(\mathrm{U})$. □

We are now going to globalize this construction. Let L be a holomorphic line bundle on X, suppose that U is such that $L|_U$ can be trivialized and fix such a trivialization. It induces a **C**-linear map $H^0(X, L) \to \mathscr{O}(U)$ which to a global section s associates the element $s_U \in \mathscr{O}(U)$ corresponding to s under our choice of trivialization. By composing this map with $d_U^{[p]}$, for $0 \leqslant p \leqslant k$, one obtains a **C**-linear map

$$d_U^{[p]} : H^0(X, L) \to \mathscr{O}\big(p_k^{-1}(U)\big)$$
$$s \mapsto d_U^{[p]} s := d_U^{[p]} s_U.$$

This map of course depends on our choice of trivialization, and whenever this map is used, it will be implicitly assumed that such a trivialization has been chosen. This should not lead to any confusion.

We may now define the Wronskian of global sections $s_0, \dots, s_k \in H^0(X, L)$ (above U with respect to our choice of trivialization) by

$$W_U(s_0, \dots, s_k) := W_U(s_{0,U}, \dots, s_{k,U}) \in E_{k,k'} T_X^*(U).$$

The Wronskian satisfies the following essential property.

Proposition **2.3.** — *For any* $s_0, \dots, s_k \in H^0(X, L)$*, the locally defined jet differential equations* $W_U(s_0, \dots, s_k)$ *glue together into a section*

$$W(s_0, \dots, s_k) \in H^0\big(X, E_{k,k'} T_X^* \otimes L^{k+1}\big).$$

Proof. — Consider open subsets $U_1, U_2 \subset X$ on which L is trivialized and let $g \in \mathscr{O}(U_{12})^*$ be the transition function from U_2 to U_1 (with $U_{12} = U_1 \cap U_2$). By definition, this means that for any $s \in H^0(X, L)$, we have

$$s_{U_1} = g s_{U_2} \in \mathscr{O}(U_{12}).$$

Applying the generalized Leibniz rule to this relation, one obtains, for each $0 \leqslant p \leqslant k$,

$$\begin{aligned} d_{U_1}^{[p]} s = d_{U_{12}}^{[p]} s_{U_1} = d_{U_{12}}^{[p]} g s_{U_2} &= \sum_{i=0}^{p} \binom{p}{i} d_{U_{12}}^{[p-i]} g d_{U_{12}}^{[i]} s_{U_2} \\ &= g d_{U_2}^{[p]} s + \sum_{i=0}^{p-1} \binom{p}{i} d_{U_{12}}^{[p-i]} g d_{U_2}^{[i]} s, \end{aligned}$$

where all the functions of this computation are restricted to $p_k^{-1}(U_{12})$. It then suffices to apply this formula to $s = s_0, \dots, s_k$ and to perform elementary operations on the lines in (6) in order to obtain

$$\mathrm{W}_{\mathrm{U}_1}(s_0,\dots,s_k)=g^{k+1}\mathrm{W}_{\mathrm{U}_2}(s_0,\dots,s_k)$$

over $p_k^{-1}(\mathrm{U}_{12})$, whence the result. □

Observe that by applying the Leibniz rule the same way as in the preceding proof, one also obtains that if A is any line bundle on X, then for any $s_0,\dots,s_k\in\mathrm{H}^0(\mathrm{X},\mathrm{L})$ and any $s\in\mathrm{H}^0(\mathrm{X},\mathrm{A})$,

$$\mathrm{W}(s\cdot s_0,\dots,s\cdot s_k)=s^{k+1}\mathrm{W}(s_0,\dots,s_k)\in\mathrm{H}^0\big(\mathrm{X},\mathrm{E}_{k,k'}\mathrm{T}^*_{\mathrm{X}}\otimes\mathrm{L}^{k+1}\otimes\mathrm{A}^{k+1}\big). \quad (\mathbf{8})$$

2.3. *The Wronskian ideal sheaf.* — Take a directed manifold (X, V), where X is a quasi-projective non-singular variety. Since $\mathrm{J}_k\mathrm{V}$ is a subbundle of $\mathrm{J}_k\mathrm{X}$, we obtain, for any $k,m\in\mathbf{N}$, a restriction morphism

$$\mathrm{res}_{\mathrm{V}}:\mathrm{E}_{k,m}\mathrm{T}^*_{\mathrm{X}}\to\mathrm{E}_{k,m}\mathrm{V}^*.$$

Therefore, for any line bundle L on X and any $s_0,\dots,s_k\in\mathrm{H}^0(\mathrm{X},\mathrm{L})$, one obtains a section

$$\mathrm{W}^{\mathrm{V}}(s_0,\dots,s_k):=\mathrm{res}_{\mathrm{V}}\big(\mathrm{W}(s_0,\dots,s_k)\big)\in\mathrm{H}^0\big(\mathrm{X},\mathrm{E}_{k,k'}\mathrm{V}^*\otimes\mathrm{L}^{k+1}\big).$$

The corresponding element under the isomorphism (3) will be denoted by

$$\omega^{\mathrm{V}}(s_0,\dots,s_k)\in\mathrm{H}^0\big(\mathrm{P}_k\mathrm{V},\mathscr{O}_{\mathrm{P}_k\mathrm{V}}(k')\otimes\pi^*_{0,k}\mathrm{L}^{k+1}\big).$$

When no confusion can arise we denote it by $\omega(s_0,\dots,s_k)$. In the relative case, we will also use the notation $\omega^{\mathrm{rel}}(s_0,\dots,s_k)$. Set

$$\begin{aligned}\mathbf{W}(\mathrm{P}_k\mathrm{V},\mathrm{L})&:=\mathrm{Span}\big\{\omega(s_0,\dots,s_k)\ \big|\ s_0,\dots,s_k\in\mathrm{H}^0(\mathrm{X},\mathrm{L})\big\}\\&\subset\mathrm{H}^0\big(\mathrm{P}_k\mathrm{V},\mathscr{O}_{\mathrm{P}_k\mathrm{V}}(k')\otimes\pi^*_{0,k}\mathrm{L}^{k+1}\big),\end{aligned}$$

and define the *kth Wronskian ideal sheaf of* L to be the ideal sheaf defined by $\mathbf{W}(\mathrm{P}_k\mathrm{V},\mathrm{L})$, which shall be denoted by $\mathfrak{w}(\mathrm{P}_k\mathrm{V},\mathrm{L})$. Recall that this means that if one considers the evaluation map

$$\mathrm{ev}:\mathbf{W}(\mathrm{P}_k\mathrm{V},\mathrm{L})\to\mathscr{O}_{\mathrm{P}_k\mathrm{V}}(k')\otimes\pi^*_{0,k}\mathrm{L}^{k+1}$$

then

$$\mathfrak{w}(\mathrm{P}_k\mathrm{V},\mathrm{L}):=\mathrm{im}(\mathrm{ev})\otimes\big(\mathscr{O}_{\mathrm{P}_k\mathrm{V}}(k')\otimes\pi^*_{0,k}\mathrm{L}^{k+1}\big)^{-1}\subset\mathscr{O}_{\mathrm{P}_k\mathrm{V}}.$$

Let us first explain that under a strong positivity hypothesis on L one can control $\mathrm{Supp}(\mathscr{O}_{\mathrm{P}_k\mathrm{V}}/\mathfrak{w}(\mathrm{P}_k\mathrm{V},\mathrm{L}))$.

Recall that one says that L separates k-jets at a point $x\in\mathrm{X}$ if the evaluation map

$$\mathrm{H}^0(\mathrm{X},\mathrm{L})\to\mathrm{L}\otimes\mathscr{O}_{\mathrm{X},x}/\mathfrak{m}^{k+1}_{\mathrm{X},x}$$

is surjective. We have the following.

Lemma **2.4.** — *If* L *separates* k*-jets at every point of* X *then*

$$\operatorname{Supp}\big(\mathscr{O}_{P_kV}/\mathfrak{w}(P_kV, L)\big) = P_kV^{\text{sing}}.$$

Proof. — Let us first prove that the left hand side is included in the right hand side. Since $P_kV^{\text{reg}} \cong J_k^{\text{reg}}V/\mathbf{G}_k \subset J_k^{\text{reg}}X/\mathbf{G}_k$, in view of (3), it suffices to show that for any $[\gamma]_k \in J_k^{\text{reg}}X$, there exist $s_0, \dots, s_k \in H^0(X, L)$ such that $W(s_0, \dots, s_k)([\gamma]_k) \neq 0$.

Take a regular k-jet $[\gamma]_k \in J_k^{\text{reg}}X$ and a neighborhood of $x := \gamma(0)$ with coordinates $(z_1, \dots, z_n)$ centered at x such that $\gamma(t) = (t, 0, \dots, 0)$ for all t in a neighborhood of 0. Take a trivialization of L over U, and global sections $s_0, \dots, s_k \in H^0(X, L)$ extending the elements $1, z_1, \frac{z_1^2}{2}, \dots, \frac{z_1^k}{k!} \in \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}^{k+1} \cong L \otimes \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}^{k+1}$. One immediately checks that $W_U(1, z_1, \dots, \frac{z_1^k}{k!})([\gamma]_k) = 1$, hence $W(s_0, \dots, s_k)([\gamma]_k) \neq 0$.

Let us now prove the reverse inclusion. Take $s_0, \dots, s_k \in H^0(X, L)$. For any germ γ satisfying $\gamma_{[k]}(0) = w_0 \in X_k^{\text{sing}}$, one has $\gamma'(0) = 0$. Hence $W(s_0, \dots, s_k)([\gamma]_k) = 0$. Indeed, the first line in the matrix of which we take the determinant vanishes. The definition of $\omega(s_0, \dots, s_k)$ then implies that $\omega(s_0, \dots, s_k)(w_0) = 0$. Since this holds for any $s_0, \dots, s_k \in H^0(X, L)$, this concludes the proof. □

Remark **2.5.** — This lemma proves that if L separates k-jets, then the base locus $\operatorname{Bs}(\mathscr{O}_{P_kV}(k') \otimes \pi_{0,k}^* L^{k+1}) \subset P_kV^{\text{sing}}$. In the absolute case $V = T_X$, if we had a negative twist L^{-1} instead of L^{k+1}, namely $\operatorname{Bs}(\mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{-1}) \subset X_k^{\text{sing}}$, then it would follow that X is hyperbolic by standard arguments building on Theorem 2.1. To obtain, for general hypersurfaces, such a result for a negative twist is the goal of our proof.

We will also need the following statement.

Lemma **2.6.** — *If* L *is very ample, then for any* $m \geqslant 0$, *one has*

$$\mathfrak{w}\big(P_kV, L^m\big) \subset \mathfrak{w}\big(P_kV, L^{m+1}\big).$$

Proof. — The assertion is local. Since L is very ample, X is covered by open subsets U of the form $U = (s \neq 0)$ for $s \in H^0(X, L)$. Given $s \in H^0(X, L)$, and $U = (s \neq 0)$ one obtains from (8) that for any $s_0, \dots, s_k \in H^0(X, L^m)$,

$$\omega(s \cdot s_0, \dots, s \cdot s_k) = s^{k+1}\omega(s_0, \dots, s_k) \in \mathbf{W}\big(P_kV, L^{m+1}\big),$$

where we write s^{k+1} instead of $\pi_{0,k}^* s^{k+1}$. The result follows. □

Therefore, given any very ample line bundle L on X we have a chain of inclusions

$$\mathfrak{w}(P_kV, L) \subset \mathfrak{w}\big(P_kV, L^2\big) \subset \cdots \subset \mathfrak{w}\big(P_kV, L^m\big) \subset \cdots$$

By noetherianity, this sequence eventually stabilizes, say after the integer $m_\infty(P_kV, L) \in \mathbf{N}$. Let us denote the obtained asymptotic ideal sheaf by

$$(\mathbf{9}) \qquad \mathfrak{w}_\infty(P_kV, L) := \mathfrak{w}\big(P_kV, L^m\big) \quad \text{for any } m \geqslant m_\infty(P_kV, L).$$

Remark **2.7.** — In order to obtain an effective bound on the degree of the hypersurfaces in the Kobayashi conjecture, one needs an upper bound on $m_\infty(P_kV, L)$. Ya Deng [21] was able to prove that $m_\infty(P_kV, L) \leqslant k$ if L is very ample. To establish this result, Ya Deng proves that if L separates k-jets at every point of X, then $\mathfrak{w}_\infty(P_kV, L) = \mathfrak{w}(P_kV, L)$. The result then follows since L^k separates k-jets at every point of X if L is very ample. This is further developed in [22].

It turns out that this ideal sheaf does not depend on the choice of the very ample line bundle L and is of purely local nature. To state this result, let us observe that for any $w \in P_kV$, writing $x = \pi_{0,k}(w)$, one can define the Wronskian at w of germs of functions $f_0, \dots, f_k \in \mathscr{O}_{X,x}$ by defining

$$(\mathbf{10}) \qquad \omega_w(f_0, \dots, f_k) = \omega_U(f_0, \dots, f_k) \in \mathscr{O}_{P_kV,w},$$

where U is an neighborhood of x on which every f_i is holomorphic. Here, the right hand side should be understood as the class, in the local ring, of the Wronskian corresponding to $W_U(f_0, \dots, f_k)$ under the isomorphism (3) and a fixed choice of trivialization of $\mathscr{O}_{P_kV}(1)$ in a neigbhorhood of w. With this notation one has the following.

Lemma **2.8.** — *Let* L *be a very ample line bundle on* X*. For any $x \in X$ and any $w \in P_kV$ such that $\pi_{0,k}(w) = x$, one has*

$$\mathfrak{w}_\infty(P_kV, L)_w = \big(\omega_w(f_0, \dots, f_k)\big)_{f_0,\dots,f_k \in \mathscr{O}_{X,x}} \subset \mathscr{O}_{P_kV,w},$$

where the right hand side denotes the ideal spanned by $\{\omega_w(f_0, \dots, f_k) \mid f_0, \dots, f_k \in \mathscr{O}_{X,x}\}$.

Proof. — That the left hand side is included in the right hand side is obvious. For the other direction, take $x \in X$, take an open neighborhood of x with holomorphic coordinates $(z_1, \dots, z_n)$ and a trivialization for $L|_U$ such that $1, z_1, \dots, z_n \in \mathscr{O}(U) \cong H^0(U, L|_U)$ all extend to global sections of $H^0(X, L)$. This is possible since L is very ample. It implies that for any $J = (j_1, \dots, j_n) \in \mathbf{N}^n$, $z^J := z_1^{j_1} \cdots z_n^{j_n}$ extends to a section in $H^0(X, L^m)$ for any $m \geqslant |J|$, so that in particular for any $P_0, \dots, P_k \in \mathbf{C}[z_1, \dots, z_n]$,

$$(\mathbf{11}) \qquad \omega_w(P_0, \dots, P_k) \in \mathfrak{w}_\infty(P_kV, L)_w.$$

Observe that if $U' \subset U$ is a neighborhood of x, $g \in \mathscr{O}(U')$, $m \geqslant 0$, $1 \leqslant i \leqslant n$ and $0 \leqslant p \leqslant k$, then there exists $\tilde{g} \in \mathscr{O}(p_k^{-1}(U'))$ such that

$$d_U^{[p]}\big(z_i^{m+k} g\big) = z_i^{m+k-p}\tilde{g} = z_i^m\big(z_i^{k-p}\tilde{g}\big),$$

where we write, by abuse of notation, $z_i := \pi_{0,k}^* z_i$. In particular, working at the level of germs, from the definition of ω_w and the multilinearity of the determinant, one obtains that for any $g_0, \dots, g_k \in \mathscr{O}_{X,x}$ there exists $q \in \mathscr{O}_{P_k V, w}$ such that

$$\omega_w\big(z_i^{m+k} g_0, g_1, \dots, g_k\big) = z_i^m q \in \mathfrak{m}^m_{P_k V, w}. \tag{12}$$

Take $f_0, \dots, f_k \in \mathscr{O}_{X,x}$. Since for any $m \geqslant 0$ and any $1 \leqslant i \leqslant n$ one can write $f_i = P_i + g_i$ with $P_i \in \mathbf{C}[z_1, \dots, z_n]$ and $g_i \in \mathfrak{m}^{n(m+k)}_{X,x}$, it follows from (11) and (12) that

$$\omega_w(f_0, \dots, f_k) \in \mathfrak{w}_\infty(P_k V, L)_w + \mathfrak{m}^m_{P_k V, w}.$$

Since this holds for any $m \in \mathbf{N}$, it follows from Krull's intersection theorem that $\omega_w(f_0, \dots, f_k) \in \mathfrak{w}_\infty(P_k V, L)_w$. □

This lemma allows us to define the *asymptotic Wronskian ideal sheaf of* $P_k V$ by

$$\mathfrak{w}_\infty(P_k V) := \mathfrak{w}_\infty(P_k V, L) \subset \mathscr{O}_{P_k V},$$

where L is any very ample line bundle on X. Moreover, if $U \subset X$ is an open subset of X, we will also set $\mathfrak{w}_\infty(P_k V|_U) := \mathfrak{w}_\infty(P_k V)|_{\pi_{0,k}^{-1}(U)}$, this is an ideal sheaf on $\pi_{0,k}^{-1}(U) = P_k V|_U$.

Lemma 2.8 also implies that $\mathfrak{w}_\infty$ behaves well under restriction.

Proposition **2.9.** — *For any* (Y, V_Y) *and* (X, V_X) *such that* $Y \subset X$ *and such that* $V_Y \subset V_X|_Y$, *under the induced inclusion* $P_k V_Y \subset P_k V_X$, *one has*

$$\mathfrak{w}_\infty(P_k V_X)|_{P_k V_Y} = \mathfrak{w}_\infty(P_k V_Y).$$

2.4. *Blow-up of the Wronskian ideal sheaf.* — The Wronskian sections defined in Section 2.2 can certainly not be used as such to apply Theorem 2.1 because of the (positive) twist by L^{k+1}. However they will be the building blocs for the jet differential equations we are going to construct. As a consequence, the ideal sheaf $\mathfrak{w}_\infty$ will be an obstruction to the positivity result (on a suitable tautological line bundle on the Demailly-Semple jet tower) we aim at. Therefore, we are led to blow up this Wronskian ideal sheaf.

Take a directed manifold (X, V), where X is a quasi-projective variety (or a Euclidian open subset of a quasi-projective variety). With the above notation, define

$$\hat{P}_k V := \mathrm{Bl}_{\mathfrak{w}_\infty(P_k V)}(P_k V) \xrightarrow{\nu_k} P_k V \tag{13}$$

to be the blow-up of $P_k V$ along $\mathfrak{w}_\infty(P_k V)$. In the absolute case we will write $\hat{X}_k = \hat{P}_k T_X$, and in the relative case we will write $\hat{\mathscr{X}}_k^{\mathrm{rel}} = \hat{P}_k T_{\mathscr{X}/T}$. A priori, we do not have any control on the singularities of $\hat{P}_k V$. Let us denote by F the effective Cartier divisor on $\hat{P}_k V$ such that

$$\mathscr{O}_{\hat{P}_k V}(-F) = \nu_k^{-1} \mathfrak{w}_\infty(P_k V) = \mathfrak{w}_\infty(P_k V) \cdot \mathscr{O}_{\hat{P}_k V}.$$

By the definition of $\mathfrak{w}_\infty(\mathrm{P}_k\mathrm{V})$ and in view of Lemma 2.8, for any very ample line bundle L on X, any $m \geqslant 0$ and any $s_0, \dots, s_k \in \mathrm{H}^0(\mathrm{X}, \mathrm{L}^m)$, there exists

$$\hat{\omega}(s_0, \dots, s_k) \in \mathrm{H}^0\big(\hat{\mathrm{P}}_k\mathrm{V}, \nu_k^*\big(\mathscr{O}_{\mathrm{P}_k\mathrm{V}}\big(k'\big) \otimes \pi_{0,k}^*\mathrm{L}^{m(k+1)}\big) \otimes \mathscr{O}_{\hat{\mathrm{P}}_k\mathrm{V}}(-\mathrm{F})\big)$$

such that, if one denotes by F$\cdot$ the map induced by the inclusion $\mathscr{O}_{\hat{\mathrm{P}}_k\mathrm{V}}(-\mathrm{F}) \to \mathscr{O}_{\hat{\mathrm{P}}_k\mathrm{V}}$,

$$\nu_k^*\omega(s_0, \dots, s_k) = \mathrm{F} \cdot \hat{\omega}(s_0, \dots, s_k). \tag{14}$$

Moreover, for any $m \geqslant m_\infty(\mathrm{P}_k\mathrm{V}, \mathrm{L})$ and any $\hat{w} \in \hat{\mathrm{P}}_k\mathrm{V}$ there exist $s_0, \dots, s_k \in \mathrm{H}^0(\mathrm{X}, \mathrm{L}^m)$ such that

$$\hat{\omega}(s_0, \dots, s_k)(\hat{w}) \neq 0. \tag{15}$$

Observe that from Proposition 2.9 one can deduce a functoriality property for these blow-ups.

Proposition **2.10.** — *For any* $(\mathrm{Y}, \mathrm{V}_\mathrm{Y}) \subset (\mathrm{X}, \mathrm{V}_\mathrm{X})$*, the inclusion* $\mathrm{P}_k\mathrm{V}_\mathrm{Y} \subset \mathrm{P}_k\mathrm{V}_\mathrm{X}$ *induces an inclusion*

$$\hat{\mathrm{P}}_k\mathrm{V}_\mathrm{Y} \subset \hat{\mathrm{P}}_k\mathrm{V}_\mathrm{X}.$$

Moreover, $\hat{\mathrm{P}}_k\mathrm{V}_\mathrm{Y}$ *is the strict transform of* $\mathrm{P}_k\mathrm{V}_\mathrm{Y}$ *in* $\hat{\mathrm{P}}_k\mathrm{V}_\mathrm{X}$ *and* $\mathscr{O}_{\hat{\mathrm{P}}_k\mathrm{V}_\mathrm{Y}}(-\mathrm{F}) = \mathscr{O}_{\hat{\mathrm{P}}_k\mathrm{V}_\mathrm{X}}(-\mathrm{F})|_{\hat{\mathrm{P}}_k\mathrm{V}_\mathrm{Y}}$.

An important consequence of Lemma 2.8 is that this blow-up process behaves well in families.

Proposition **2.11.** — *Let* $\mathscr{X} \xrightarrow{\rho} \mathrm{T}$ *be a smooth and projective morphism between non-singular quasi-projective varieties. Take* $\nu_k : \hat{\mathscr{X}}_k^{\mathrm{rel}} \to \mathscr{X}_k^{\mathrm{rel}}$ *as above and consider* $\rho_k = \rho \circ \pi_{0,k} : \mathscr{X}_k^{\mathrm{rel}} \to \mathrm{T}$*. For any* $t_0 \in \mathrm{T}$*, writing* $\mathrm{X}_{t_0} := \rho^{-1}(t_0)$ *and using the isomorphism* $\mathrm{X}_{t_0,k} = \rho_k^{-1}(t_0) \subset \mathscr{X}_k^{\mathrm{rel}}$*, one has*

$$\nu_k^{-1}(\mathrm{X}_{t_0,k}) = \hat{\mathrm{X}}_{t_0,k} \quad \textit{and} \quad \mathscr{O}_{\hat{\mathscr{X}}_k^{\mathrm{rel}}}(-\mathrm{F})|_{\nu_k^{-1}(\mathrm{X}_{t_0,k})} \cong \mathscr{O}_{\hat{\mathrm{X}}_{t_0,k}}(-\mathrm{F}).$$

Proof. — The key point of the argument is to prove that the family under consideration with the Wronskian ideal sheaf is locally a product.

Take $t_0 \in \mathrm{T}$ and $x \in \mathrm{X}_{t_0} \subset \mathscr{X}$. Take a neighborhood $\mathrm{U} \subset \mathscr{X}$ of x such that $\mathrm{U} \cong \mathrm{U}_1 \times \mathrm{U}_2$ where $\mathrm{U}_1 \subset \mathrm{T}$ is a neighborhood of t_0 and where $\mathrm{U}_2 \subset \mathbf{C}^n$ and such that under this isomorphism, the map ρ is identified with the first projection $p_1 : \mathrm{U} \to \mathrm{U}_1$. This can be achieved since ρ is a smooth morphism. Denoting by $p_2 : \mathrm{U} \to \mathrm{U}_2$ the second projection, one obtains an isomorphism

$$\pi_{0,k}^{-1}(\mathrm{U}) = \mathrm{P}_k\mathrm{T}_{\mathrm{U}/\mathrm{T}} \cong \mathrm{U}_1 \times \mathrm{P}_k\mathrm{T}_{\mathrm{U}_2}.$$

Composing it with the second projection, one obtains a morphism $p_2^k : \mathrm{P}_k\mathrm{T}_{\mathrm{U}/\mathrm{T}} \to \mathrm{P}_k\mathrm{T}_{\mathrm{U}_2}$. We are going to prove that

$$\mathfrak{w}_\infty(\mathrm{P}_k\mathrm{T}_{\mathrm{U}/\mathrm{T}}) = \big(p_2^k\big)^{-1}\mathfrak{w}_\infty(\mathrm{P}_k\mathrm{T}_{\mathrm{U}_2}). \tag{16}$$

Since $\mathfrak{w}_\infty(\mathrm{P}_k\mathrm{T}_{\mathrm{U}/\mathrm{T}}) = \mathfrak{w}_\infty(\mathscr{X}_k^{\mathrm{rel}})|_{\pi_{0,k}^{-1}(\mathrm{U})}$, this will conclude the proof. Indeed, this will imply that

$$(\pi_{0,k}\circ \nu_k)^{-1}(\mathrm{U}) = \hat{\mathrm{P}}_k\mathrm{T}_{\mathrm{U}/\mathrm{T}} \cong \mathrm{U}_1 \times \hat{\mathrm{P}}_k\mathrm{T}_{\mathrm{U}_2},$$

and since moreover $\hat{\mathrm{X}}_{t_0,k} \cap (\pi_{0,k}\circ\nu_k)^{-1}(\mathrm{U}) \cong \hat{\mathrm{P}}_k\mathrm{T}_{\mathrm{U}_2}$, the result follows.

To prove (16), we take $w \in \pi_{0,k}^{-1}(\mathrm{U}) = \mathrm{P}_k\mathrm{T}_{\mathrm{U}/\mathrm{T}}$ and prove the desired equality at the level of stalks at w. Set $x = \pi_{0,k}(w)$. From Lemma 2.8, it follows that $\mathfrak{w}_\infty(\mathrm{P}_k\mathrm{T}_{\mathrm{U}/\mathrm{T}})_w$ is spanned by the Wronskians of the form $\omega_w^{\mathrm{rel}}(f_0,\dots,f_k)$ where $f_0,\dots,f_k \in \mathscr{O}_{\mathscr{X},x}$, and that $\mathfrak{w}_\infty(\mathrm{P}_k\mathrm{T}_{\mathrm{U}_2})_{p_2^k(w)}$ is spanned by Wronskians of the form $\omega_{p_2^k(x)}(g_0,\dots,g_k)$ where $g_0,\dots,g_k \in \mathscr{O}_{\mathrm{U}_2,p_2(x)}$. Observe that for any $g_0,\dots,g_k \in \mathscr{O}_{\mathrm{U}_2,p_2(x)}$, one has

$$\big(p_{2,w}^k\big)^*\omega_{p_2^k(w)}(g_0,\dots,g_k) = u\,\omega_w^{\mathrm{rel}}\big(p_{2,x}^*g_0,\dots,p_{2,x}^*g_k\big) \in \mathscr{O}_{\mathscr{X}_k^{\mathrm{rel}},w}.$$

Here $(p_{2,w}^k)^* : \mathscr{O}_{\mathrm{P}_k\mathrm{T}_{\mathrm{U}_2},p_2^k(w)} \to \mathscr{O}_{\mathrm{P}_k\mathrm{T}_{\mathrm{U}/\mathrm{T}},w}$ and $p_{2,x}^* : \mathscr{O}_{\mathrm{U}_2,p_2(x)} \to \mathscr{O}_{\mathrm{U},x}$ are induced by p_2^k and p_2, and $u \in \mathscr{O}_{\mathrm{X}_k,w}^*$ is a unit depending on the choice of trivialization used in (10). This proves already that the left hand side of (16) contains the right hand side. Take coordinates $(\underline{t})$ centered at $\rho(x) = p_1(x) \in \mathrm{U}_1$ and coordinates $(\underline{z})$ centered at $p_2(x) \in \mathrm{U}_2$. These induce coordinates $(\underline{t},\underline{z})$ on U centered at x. Observe that for any $\mathrm{I}_0,\dots,\mathrm{I}_k \in \mathbf{N}^{\dim \mathrm{T}}$ and any $\mathrm{J}_0,\dots,\mathrm{J}_k \in \mathbf{N}^n$, one has

$$\begin{aligned}\omega_w^{\mathrm{rel}}\big(t^{\mathrm{I}_0}z^{\mathrm{J}_0},\dots,t^{\mathrm{I}_k}z^{\mathrm{J}_k}\big) &= t^{\mathrm{I}_0+\cdots+\mathrm{I}_k}\omega_w^{\mathrm{rel}}\big(z^{\mathrm{J}_0},\dots,z^{\mathrm{J}_k}\big)\\ &= u^{-1}t^{\mathrm{I}_0+\cdots+\mathrm{I}_k}\big(p_{2,w}^k\big)^*\omega_{p_2^k(w)}\big(z^{\mathrm{J}_0},\dots,z^{\mathrm{J}_k}\big).\end{aligned} \tag{17}$$

This follows from the fact that the computation takes place in the relative jet space, so that one can consider $t_1,\dots,t_{\dim \mathrm{T}}$ as constants, from which the formula follows by multilinearity. This implies in particular that for any $\mathrm{P}_0,\dots,\mathrm{P}_k \in \mathbf{C}[\underline{t},\underline{z}]$,

$$\omega_w(\mathrm{P}_0,\dots,\mathrm{P}_k) \in \big(p_{2,w}^k\big)^{-1}\big(\mathfrak{w}_\infty(\mathrm{P}_k\mathrm{T}_{\mathrm{U}_2})_{p_2^k(w)}\big) = \Big(\big(p_{2,w}^k\big)^{-1}\mathfrak{w}_\infty(\mathrm{P}_k\mathrm{T}_{\mathrm{U}_2})\Big)_w.$$

From this, (16) follows from Krull's intersection theorem, as in the proof of Lemma 2.8. □

Remark **2.12.** — Let us mention that, as was pointed out to us by O. Benoist, if we take $\hat{\mathrm{P}}_k\mathrm{V} \to \mathrm{P}_k\mathrm{V}$ to be a resolution of the Wronskian ideal sheaf obtained by using a resolution algorithm that commutes with smooth morphisms in the analytic category (as constructed in [36]), then Proposition 2.11 would still be valid. With this at hand, one could make the rest of the paper with this definition, this would not change anything except that the proof of the implication *Theorem 3.2* ⇒ *Main Theorem* in Section 3.1 below would be slightly more involved. While this would allow us to work only with non-singular varieties, we prefer to use the more elementary definition of $\hat{\mathrm{P}}_k\mathrm{V}$ above.

A key point in the proof of the main theorem is the use of a property which is strictly stronger than hyperbolicity and which is Zariski open. This is precisely condition $(*)$ in the following proposition.

Proposition **2.13.** — *Let* X *be a smooth projective variety. If*

$$(*) \qquad \exists a_1, \dots, a_k, q \in \mathbf{N} \quad \text{such that} \quad \nu_k^* \mathscr{O}_{X_k}(a_k, \dots, a_1) \otimes \mathscr{O}_{\hat{X}_k}(-qF) \quad \text{is ample},$$

then X *is hyperbolic. Moreover, property* $(*)$ *is a Zariski open property. Namely, given a smooth projective morphism* $\mathscr{X} \xrightarrow{\rho} T$ *between quasi-projective varieties, if there exists* $t_0 \in T$ *such that* X_{t_0} *satisfies* $(*)$ *then, for general* $t \in T$, X_t *satisfies* $(*)$.

Proof. — If $(*)$ is satisfied, then one can find integers $b_1, \dots, b_k, s \in \mathbf{N}$ and an ample line bundle A on X such that

$$\nu_k^*\big(\mathscr{O}_{X_k}(b_k, \dots, b_1) \otimes \pi_{0,k}^* A^{-1}\big) \otimes \mathscr{O}_{\hat{X}_k}(-sF)$$

is base point free. From this one sees that multiplication by sF induces a linear map

$$\begin{aligned} &H^0\big(\hat{X}_k, \nu_k^*\big(\mathscr{O}_{X_k}(b_k, \dots, b_1) \otimes \pi_{0,k}^* A^{-1}\big) \otimes \mathscr{O}_{\hat{X}_k}(-sF)\big) \\ &\quad \xrightarrow{\cdot sF} H^0\big(\hat{X}_k, \nu_k^*\big(\mathscr{O}_{X_k}(b_k, \dots, b_1) \otimes \pi_{0,k}^* A^{-1}\big)\big), \end{aligned}$$

which defines a linear system $S := \mathrm{im}(\cdot sF)$ whose base locus $\mathrm{Bs}(S)$ is included (set-theoretically) in $\mathrm{Supp}(F)$. But this implies, by Lemma 2.4, that the induced linear system

$$(\nu_k)_* S \subset H^0\big(X_k, \mathscr{O}_{X_k}(b_k, \dots, b_1) \otimes \pi_{0,k}^* A^{-1}\big)$$

satisfies $\mathrm{Bs}((\nu_k)_* S) \subset \mathrm{Supp}(\mathscr{O}_{X_k}/\mathfrak{w}_\infty(X_k)) \subset X_k^{\mathrm{sing}}$. In particular, one has

$$(\mathbf{18}) \qquad \mathrm{Bs}\big(\mathscr{O}_{X_k}(b_k, \dots, b_1) \otimes \pi_{0,k}^* A^{-1}\big) \subset X_k^{\mathrm{sing}}.$$

Now, if $f : \mathbf{C} \to X$ is a non-constant entire curve, then Theorem 2.1 (with $V = T_X$) implies that

$$f_{[k]}(\mathbf{C}) \subset \mathrm{Bs}\big(\mathscr{O}_{X_k}(b_k, \dots, b_1) \otimes \pi_{0,k}^* A^{-1}\big) \subset X_k^{\mathrm{sing}},$$

which is a impossible since f is non-constant. From this one deduces that X is hyperbolic. The second part of the statement, about the Zariski openness, follows immediately from Proposition 2.11 and the openness property of ampleness. □

Remark **2.14.** — Let us mention that this argument actually proves that condition $(*)$ implies that the Green-Griffiths locus of X, as defined in [26], is empty. Indeed, this is a direct consequence of (18). In view of Theorem 2.1, this last condition is well known to imply hyperbolicity (by the above argument), and it is in fact a strictly stronger condition, as is explained in [26]. In the present paper we prove that general sufficiently ample hypersurfaces satisfy condition $(*)$.

3. Proof of the main theorem

3.1. *Setting.* — Let us introduce the framework in which we will work from now on. Let X be a smooth n-dimensional projective variety and let A be an ample line bundle on X. Fix integers N, k such that $\mathrm{N} \geqslant n \geqslant 2$ and $k \geqslant \mathrm{N}-1$. The integer N should be thought of as the number of "variables", and the integer k as the jet order.

Let us emphasize that in order to prove the main theorem, we could restrict ourselves to the case $\mathrm{N}=n$ and $k=n-1$. Nevertheless, in view of possible further developments, we work in a slightly greater generality.

Take $\nu_k : \hat{\mathrm{X}}_k \to \mathrm{X}_k$ and $\mathscr{O}_{\hat{\mathrm{X}}_k}(-\mathrm{F})$ as in Section 2.4. Take $v_0 \in \mathbf{N}$ such that A^v is very ample for any $v \geqslant v_0$. Fix two integers $v, u \geqslant v_0$. The reader interested in the case when A is very ample can take $v_0 = v = u = 1$ in the rest of this article. Let us now fix $\tau_0, \ldots, \tau_\mathrm{N} \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^v)$ in general position. Let us assume moreover that, denoting by $\mathrm{C}_1, \ldots, \mathrm{C}_\kappa$ the irreducible and embedded components of the scheme defined by $\mathfrak{w}_\infty(\mathrm{X}_k)$, one has

$$\mathrm{C}_j \not\subseteq \left(\pi_{0,k}^* \tau_i = 0\right) \quad \forall 1 \leqslant j \leqslant \kappa,\ \forall 0 \leqslant i \leqslant \mathrm{N}. \tag{19}$$

One can make this assumption since A^v is very ample. To ease our notation, we will from now on write τ_i instead of $\pi_{0,k}^* \tau_i$ when no confusion can arise.

Remark **3.1.** — Let us mention that one can actually prove that for any $1 \leqslant j \leqslant \kappa$, one has $\pi_{0,k}(\mathrm{C}_j) = \mathrm{X}$. Therefore (19) is always satisfied. This can be proven using Lemma 2.8 and the Noether-Lasker decomposition for analytic coherent sheaves as established in [57]. Since we do not need this refinement to prove our main result, we take (19) as a hypothesis.

Fix also integers $\varepsilon, \delta, r \geqslant 1$. Set $\mathbf{I} := \{\mathrm{I} = (i_0, \ldots, i_\mathrm{N}) \in \mathbf{N}^{n+1} \mid |\mathrm{I}| = \delta\}$. We are going to focus on hypersurfaces of X defined by sections of the form

$$\mathrm{F}(\mathbf{a}) := \sum_{\mathrm{I} \in \mathbf{I}} a_\mathrm{I} \tau^{(r+k)\mathrm{I}} \in \mathrm{H}^0\big(\mathrm{X}, \mathrm{A}^{u\varepsilon + (r+k)v\delta}\big), \tag{20}$$

where for all $\mathrm{I} \in \mathbf{I}$, $a_\mathrm{I} \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{u\varepsilon})$, so that $\mathbf{a} := (a_\mathrm{I})_{\mathrm{I} \in \mathbf{I}} \in \mathbf{A} := \bigoplus_{\mathrm{I} \in \mathbf{I}} \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{u\varepsilon})$. Here we used the multi-index notation $\tau^\mathrm{I} = \tau_0^{i_0} \cdots \tau_\mathrm{N}^{i_\mathrm{N}}$ for $\mathrm{I} = (i_0, \ldots, i_\mathrm{N})$. Consider the universal family

$$\mathscr{H} := \big\{(\mathbf{a}, x) \in \mathbf{A} \times \mathrm{X} \;\big|\; \mathrm{F}(\mathbf{a})(x) = 0\big\}.$$

Let us denote by $\rho : \mathscr{H} \to \mathbf{A}$ the natural projection. For any $\mathbf{a} \in \mathbf{A}$, set $\mathrm{H}_\mathbf{a} := \rho^{-1}(\mathbf{a})$, and let us consider the smooth locus $\mathbf{A}_{\mathrm{sm}} := \{\mathbf{a} \in \mathbf{A} \mid \mathrm{H}_\mathbf{a} \text{ is smooth}\}$ which is a non-empty Zariski open subset of $\mathbf{A}$. Let us also denote by $\rho : \mathscr{H} \to \mathbf{A}_{\mathrm{sm}}$ the restricted family. Consider the relative kth order jet space $\mathscr{H}_k^{\mathrm{rel}} \to \mathscr{H}$ and set $\rho_k := \rho \circ \pi_{0,k} : \mathscr{H}_k^{\mathrm{rel}} \to \mathbf{A}_{\mathrm{sm}}$.

Consider also the blow-up of the Wronskian ideal sheaf $\hat{\mathscr{H}}_k^{\mathrm{rel}} \to \mathscr{H}_k^{\mathrm{rel}}$ and set $\hat{\rho}_k := \rho_k \circ \nu_k : \hat{\mathscr{H}}_k^{\mathrm{rel}} \to \mathbf{A}_{\mathrm{sm}}$. It follows from Proposition 2.10 that $\hat{\mathscr{H}}_k^{\mathrm{rel}} \subset \mathbf{A}_{\mathrm{sm}} \times \hat{\mathrm{X}}_k$. For any $\mathbf{a} \in \mathbf{A}_{\mathrm{sm}}$, define $\mathrm{H}_{k,\mathbf{a}} := \rho_k^{-1}(\mathbf{a})$ and $\hat{\mathrm{H}}_{k,a} := \hat{\rho}_k^{-1}(\mathbf{a})$. The projection $\mathscr{H}_k^{\mathrm{rel}} \to \mathrm{X}_k$ induces an inclusion $\mathrm{H}_{k,\mathbf{a}} \subset \mathrm{X}_k$ and the projection $\hat{\mathscr{H}}_k^{\mathrm{rel}} \to \hat{\mathrm{X}}_k$ induces an inclusion $\hat{\mathrm{H}}_{k,\mathbf{a}} \subset \hat{\mathrm{X}}_k$.

With the notation of (9), let us set $m_\infty := m_\infty(\mathrm{X}_k, \mathrm{A}^u)$. The aim of the rest of this paper is to prove the following result.

Theorem **3.2.** — *Take $v, u \geqslant v_0$. Suppose $\mathrm{N} \geqslant n$, $k \geqslant \mathrm{N} - 1$, $\varepsilon \geqslant m_\infty$ and $\delta \geqslant n(k+1)$. There exists $\mathrm{M} = \mathrm{M}(\mathrm{N}, k, \delta) \in \mathbf{N}$, and $r(v, u, \mathrm{M}, \mathrm{N}, k, \varepsilon, \delta) \in \mathbf{N}$ such that if $r \geqslant r(v, u, \mathrm{M}, \mathrm{N}, k, \varepsilon, \delta)$, then there exists a non-empty Zariski open subset $\mathbf{A}_{\mathrm{nef}} \subset \mathbf{A}_{\mathrm{sm}}$ such that for any $\mathbf{a} \in \mathbf{A}_{\mathrm{nef}}$ the line bundle*

$$\nu_k^*\big(\mathscr{O}_{\mathrm{X}_k}(\mathrm{M}k') \otimes \pi_{0,k}^*\mathrm{A}^{-1}\big) \otimes \mathscr{O}_{\hat{\mathrm{X}}_k}(-\mathrm{MF})|_{\hat{\mathrm{H}}_{k,\mathbf{a}}}$$

is nef on $\hat{\mathrm{H}}_{k,\mathbf{a}}$.

Let us first explain how this theorem implies our main result.

Theorem 3.2 ⇒ Main Theorem. — It follows from Proposition 2.10, applied to the pairs $(\mathrm{H}_{\mathbf{a}}, \mathrm{T}_{\mathrm{H}_{\mathbf{a}}}) \subset (\mathrm{X}, \mathrm{T}_{\mathrm{X}})$ and from Proposition 2.11 applied to the family $\mathscr{H} \to \mathbf{A}_{\mathrm{sm}}$, that $\hat{\mathrm{H}}_{\mathbf{a},k} \cong \hat{\mathrm{H}}_{k,\mathbf{a}} \subset \hat{\mathrm{X}}_k$ and that moreover $\mathscr{O}_{\hat{\mathrm{X}}_k}(-\mathrm{MF})|_{\hat{\mathrm{H}}_{k,\mathbf{a}}} \cong \mathscr{O}_{\hat{\mathrm{H}}_{\mathbf{a},k}}(-\mathrm{MF})$. The conclusion of Theorem 3.2 implies, after tensoring by suitable line bundles, that $\mathrm{H}_{\mathbf{a}}$ satisfies property $(*)$ for any $\mathbf{a} \in \mathbf{A}_{\mathrm{nef}}$. By Proposition 2.13, one deduces that for v, u, ε, δ, r as above, general hypersurfaces in $|\mathrm{A}^{u\varepsilon+(r+k)v\delta}|$ satisfy property $(*)$ and are therefore hyperbolic.

To conclude the proof, it suffices to show, by adjusting the different exponents, that this gives the result for general hypersurfaces in $|\mathrm{A}^d|$ for all d sufficiently large. This can be seen as follows. Take $\delta = n(k+1)$, $v = v_0$ and $u \geqslant v_0$ such that $\gcd(u, v\delta) = 1$. Take $\mathrm{R} := \max\{r(v, u, \mathrm{M}, \mathrm{N}, k, \varepsilon, \delta) \mid m_\infty \leqslant \varepsilon < m_\infty + v\delta\}$ and set $d_0 := u(m_\infty + v\delta) + (\mathrm{R} + k)v\delta$. We will show the result holds for any $d \geqslant d_0$.

It suffices to prove that any integer $d \geqslant d_0$ can be written as $d = u\varepsilon + (r+k)v\delta$ for a convenient choice of $r \geqslant \mathrm{R}$ and $m_\infty \leqslant \varepsilon < m_\infty + v\delta$. For $d \geqslant d_0$, take ε to be the unique element in $\{m_\infty, \dots, m_\infty + v\delta - 1\}$ such that $u\varepsilon \equiv d\ [v\delta]$, which is possible since $\gcd(u, v\delta) = 1$. Then $d - u\varepsilon = tv\delta$ for some $t \in \mathbf{Z}$. But since $d \geqslant d_0$, one has $tv\delta \geqslant (\mathrm{R} + k)v\delta$, and we may take $r = t - k \geqslant \mathrm{R}$. □

Let us mention that $\mathrm{M}(\mathrm{N}, k, \delta)$ is not effective, but only depends on N, k, δ, as will be clear from its definition below formula (40). This is the second point at which we loose effectivity on the bound. We refer to Remark 3.13 for more details on this point.

3.2. *Maps to the Grassmanian.* — In this entire section, take $N \geqslant n \geqslant 2$ and $k \geqslant 1$. Note that the hypothesis on k is less restrictive than the hypothesis of Theorem 3.2. We do this in order to present the results of this section in their appropriate generality. The main idea in the proof of the positivity statement in Theorem 3.2 is to construct a map from $\hat{\mathscr{H}}_k^{\mathrm{rel}}$ to a suitable generically finite family and to use the positivity of the tautological bundle on the parameter space of this family. Before doing so, we need some preliminaries which we describe in this section. Let us start with several computational lemmata.

Lemma **3.3.** — *Let* U *be an open subset of* X *on which* A *can be trivialized, and fix such a trivialization. Take* $\mathrm{I} = (i_0, \dots, i_\mathrm{N})$. *For any* $0 \leqslant p \leqslant k$, *there exists a* $\mathbf{C}$*-linear map*

$$d_{\mathrm{I,U}}^{[p]} : \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{u\varepsilon}) \to \mathscr{O}\big(p_k^{-1}(\mathrm{U})\big)$$

such that for any $a \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{u\varepsilon})$, *we have* $d_{\mathrm{U}}^{[p]}(a\tau^{(r+k)\mathrm{I}}) = \tau_{\mathrm{U}}^{r\mathrm{I}} d_{\mathrm{I,U}}^{[p]}(a)$.

Proof. — By induction, there exists $\tilde{a}$ such that $d_{\mathrm{U}}^{[p]}(a\tau^{(r+k)\mathrm{I}}) = \tau_{\mathrm{U}}^{(r+k-p)\mathrm{I}}\tilde{a}$. It now suffices to define $d_{\mathrm{I,U}}^{[p]}(a) := \tau_{\mathrm{U}}^{(k-p)\mathrm{I}}\tilde{a}$. ☐

Given any open subset U, any trivialization of $\mathrm{A}|_{\mathrm{U}}$ as in Lemma 3.3, any $\mathrm{I}_0, \dots, \mathrm{I}_k \in \mathbf{I}$ and any $a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k} \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{u\varepsilon})$, one can define

$$\mathrm{W}_{\mathrm{U},\mathrm{I}_0,\dots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k}) := \begin{vmatrix} d_{\mathrm{I}_0,\mathrm{U}}^{[0]}(a_{\mathrm{I}_0}) & \cdots & d_{\mathrm{I}_k,\mathrm{U}}^{[0]}(a_{\mathrm{I}_k}) \\ \vdots & \ddots & \vdots \\ d_{\mathrm{I}_0,\mathrm{U}}^{[k]}(a_{\mathrm{I}_0}) & \cdots & d_{\mathrm{I}_k,\mathrm{U}}^{[k]}(a_{\mathrm{I}_k}) \end{vmatrix} \in \mathscr{O}\big(p_k^{-1}(\mathrm{U})\big). \tag{21}$$

From Lemma 3.3, it follows immediately that

$$\mathrm{W}_{\mathrm{U}}\big(a_{\mathrm{I}_0}\tau^{(r+k)\mathrm{I}_0}, \dots, a_{\mathrm{I}_k}\tau^{(r+k)\mathrm{I}_k}\big) = \tau_{\mathrm{U}}^{r(\mathrm{I}_0+\cdots+\mathrm{I}_k)}\mathrm{W}_{\mathrm{U},\mathrm{I}_0,\dots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k}).$$

Therefore, from Proposition 2.3, one deduces the following.

Lemma **3.4.** — *For any* $\mathrm{I}_0, \dots, \mathrm{I}_k \in \mathbf{I}$ *and any* $a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k} \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{u\varepsilon})$, *the locally defined functions* $\mathrm{W}_{\mathrm{U},\mathrm{I}_0,\dots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k})$ *glue together into a global section*

$$\mathrm{W}_{\mathrm{I}_0,\dots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k}) \in \mathrm{H}^0\big(\mathrm{X}, \mathrm{E}_{k,k'}\mathrm{T}_{\mathrm{X}}^* \otimes \mathrm{A}^{(k+1)(u\varepsilon+kv\delta)}\big),$$

such that $\mathrm{W}(a_{\mathrm{I}_0}\tau^{(r+k)\mathrm{I}_0}, \dots, a_{\mathrm{I}_k}\tau^{(r+k)\mathrm{I}_k}) = \tau^{r(\mathrm{I}_0+\cdots+\mathrm{I}_k)}\mathrm{W}_{\mathrm{I}_0,\dots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k})$.

Let us denote the global section induced via the isomorphism (3) by

$$\omega_{\mathrm{I}_0,\dots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k}) \in \mathrm{H}^0\big(\mathrm{X}_k, \mathscr{O}_{\mathrm{X}_k}(k') \otimes \pi_{0,k}^*\mathrm{A}^{(k+1)(u\varepsilon+kv\delta)}\big).$$

Note that the line bundle involved in that expression does not depend on r, and that

$$\omega\big(a_{\mathrm{I}_0}\tau^{(r+k)\mathrm{I}_0}, \dots, a_{\mathrm{I}_k}\tau^{(r+k)\mathrm{I}_k}\big) = \tau^{r(\mathrm{I}_0+\cdots+\mathrm{I}_k)}\omega_{\mathrm{I}_0,\dots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k}). \tag{22}$$

Consider the rational map

$$\begin{aligned}\Phi : \mathbf{A} \times X_k &\dashrightarrow P\big(\Lambda^{k+1}\mathbf{C}^{\mathbf{I}}\big)\\ (\mathbf{a}, w) &\mapsto \Big(\big[\omega_{I_0,\ldots,I_k}(a_{I_0},\ldots,a_{I_k})(w)\big]\Big)_{I_0,\ldots,I_k\in\mathbf{I}},\end{aligned}$$

where $\mathbf{C}^{\mathbf{I}} := \bigoplus_{I\in\mathbf{I}} \mathbf{C} \cong \mathbf{C}^{\binom{N+\delta}{\delta}}$. Let us prove that Φ factors through the Plücker embedding. Fix $w_0 \in X_k$, take U_{w_0} and $(\gamma_w)_{w\in U_{w_0}}$ as in (4) and U as in Lemma 3.3, such that $U_{w_0} \subset \pi_{0,k}^{-1}(U)$. For any $\mathbf{a} = (a_I)_{I\in\mathbf{I}} \in \mathbf{A}$, any $w \in U_{w_0}$ and any $0 \leqslant p \leqslant k$ let us define

$$d^{[p]}_{\bullet,w_0}(\mathbf{a}, w) := \Big(d^{[p]}_{I,U}(a_I)\big([\gamma_w]_k\big)\Big)_{I\in\mathbf{I}} \in \mathbf{C}^{\mathbf{I}}.$$

This definition depends on the choice of w_0, the choice of the family (γ_w) and the choice of the trivialization of A over U. Nevertheless, considering the rational map

$$\begin{aligned}\Phi_{w_0} : \mathbf{A} \times U_{w_0} &\dashrightarrow \mathrm{Gr}_{k+1}\big(\mathbf{C}^{\mathbf{I}}\big)\\ (\mathbf{a}, w) &\mapsto \mathrm{Span}\big(d^{[0]}_{\bullet,w_0}(\mathbf{a}, w),\ldots,d^{[k]}_{\bullet,w_0}(\mathbf{a}, w)\big),\end{aligned}\tag{23}$$

we have $\Phi|_{U_{w_0}} = \mathrm{Pluc}\circ\Phi_{w_0}$, where $\mathrm{Pluc} : \mathrm{Gr}_{k+1}(\mathbf{C}^{\mathbf{I}}) \hookrightarrow P(\Lambda^{k+1}\mathbf{C}^{\mathbf{I}})$ denotes the Plücker embedding. This proves that Φ factors through the Plücker embedding and we will denote (slightly abusively) by $\Phi : \mathbf{A} \times X_k \dashrightarrow \mathrm{Gr}_{k+1}(\mathbf{C}^{\mathbf{I}})$ the induced map into the Grassmannian.

Our aim is to prove that ν_k partially resolves the indeterminacies of Φ. In order to do so, we rely on the following consequence of our hypothesis (19).

Lemma **3.5.** — *Let* L *be a line bundle on* X_k *with a global section* $\omega \in H^0(X_k, L)$. *Let* $I \in \mathbf{N}^{n+1}$. *If* $\tau^I\omega$ *satisfies*

$$\tau^I\omega \in H^0\big(X_k, L \otimes \pi_{0,k}^*A^{\nu|I|} \otimes \mathfrak{w}_\infty(X_k)\big),$$

then ω *vanishes along* $\mathfrak{w}_\infty(X_k)$, *that is to say* $\omega \in H^0(X_k, L \otimes \mathfrak{w}_\infty(X_k))$.

Proof. — We will prove the announced vanishing statement locally. Take $w \in X_k$ and let us denote by τ^I_w and ω_w the germs at w of the local representative of τ^I and ω with respect to a fixed choice of trivializations for A^ν and L. Consider the Noether-Lasker decomposition of $\mathfrak{w}_\infty(X_k)_w \subseteq \mathscr{O}_{X_k,w}$:

$$\mathfrak{w}_\infty(X_k)_w = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\mu,$$

where for every $1 \leqslant i \leqslant \mu$, $\mathfrak{q}_i$ is a primary ideal of $\mathscr{O}_{X_k,w}$. For any $1 \leqslant i \leqslant \mu$, our hypothesis implies that $\tau^I_w\omega_w \in \mathfrak{q}_i$. On the other hand, by the definition of irreducible and embedded components, our assumption (19) implies that $\tau^I_w \notin \sqrt{\mathfrak{q}_i}$. Therefore, for any $1 \leqslant i \leqslant \mu$, since $\mathfrak{q}_i$ is primary, it follows that $\omega_w \in \mathfrak{q}_i$. This implies that $\omega_w \in \mathfrak{w}_\infty(X_k)_w$, which concludes the proof. □

From (22), it follows that for any $\mathrm{I}_0, \ldots, \mathrm{I}_k \in \mathbf{I}$ and any $a_{\mathrm{I}_0}, \ldots, a_{\mathrm{I}_k} \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{u\varepsilon})$, one has

$$\tau^{r(\mathrm{I}_0+\cdots+\mathrm{I}_k)}\omega_{\mathrm{I}_0,\ldots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \ldots, a_{\mathrm{I}_k}) \\ \in \mathrm{H}^0\big(\mathrm{X}_k, \mathscr{O}_{\mathrm{X}_k}\big(k'\big) \otimes \pi_{0,k}^*\mathrm{A}^{(k+1)(u\varepsilon+(k+r)v\delta)} \otimes \mathfrak{w}_\infty(\mathrm{X}_k)\big).$$

Lemma 3.5 therefore implies that $\omega_{\mathrm{I}_0,\ldots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \ldots, a_{\mathrm{I}_k})$ vanishes along $\mathfrak{w}_\infty(\mathrm{X}_k)$:

$$\omega_{\mathrm{I}_0,\ldots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \ldots, a_{\mathrm{I}_k}) \in \mathrm{H}^0\big(\mathrm{X}_k, \mathscr{O}_{\mathrm{X}_k}\big(k'\big) \otimes \pi_{0,k}^*\mathrm{A}^{(k+1)(u\varepsilon+kv\delta)} \otimes \mathfrak{w}_\infty(\mathrm{X}_k)\big).$$

This implies the existence of a global section

$$\hat{\omega}_{\mathrm{I}_0,\ldots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \ldots, a_{\mathrm{I}_k}) \in \mathrm{H}^0\big(\hat{\mathrm{X}}_k, \nu_k^*\big(\mathscr{O}_{\mathrm{X}_k}\big(k'\big) \otimes \pi_{0,k}^*\mathrm{A}^{(k+1)(u\varepsilon+kv\delta)}\big) \otimes \mathscr{O}_{\hat{\mathrm{X}}_k}(-\mathrm{F})\big) \tag{24}$$

satisfying

$$\nu_k^*\omega_{\mathrm{I}_0,\ldots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \ldots, a_{\mathrm{I}_k}) = \mathrm{F} \cdot \hat{\omega}_{\mathrm{I}_0,\ldots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \ldots, a_{\mathrm{I}_k}).$$

From the multilinearity property of $\hat{\omega}_{\mathrm{I}_0,\ldots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \ldots, a_{\mathrm{I}_k})$, it makes sense to consider the rational map

$$\hat{\Phi} : \mathbf{A} \times \hat{\mathrm{X}}_k \dashrightarrow \mathrm{P}\big(\Lambda^{k+1}\mathbf{C}^{\mathbf{I}}\big) \\ (\mathbf{a}, \hat{w}) \mapsto \big[\big(\hat{\omega}_{\mathrm{I}_0,\ldots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \ldots, a_{\mathrm{I}_k})(\hat{w})\big)_{\mathrm{I}_0,\ldots,\mathrm{I}_k\in\mathbf{I}}\big].$$

Observe that outside Supp(F), one has $\hat{\Phi} = \Phi \circ \nu_k$. Since $\hat{\mathrm{X}}_k$ is irreducible, this implies that $\hat{\Phi}$ also factors through the Plücker embedding. We shall also denote by $\hat{\Phi}$ the obtained map

$$\hat{\Phi} : \mathbf{A} \times \hat{\mathrm{X}}_k \dashrightarrow \mathrm{Gr}_{k+1}\big(\mathbf{C}^{\mathbf{I}}\big).$$

We will need a local description for $\hat{\Phi}$ similar to (23).

Lemma **3.6.** — *Suppose* $\varepsilon \geqslant m_\infty$. *For any* $\hat{w}_0 \in \hat{\mathrm{X}}_k$ *there exists an open neighborhood* $\hat{\mathrm{U}}_{\hat{w}_0} \subset \hat{\mathrm{X}}_k$ *of* $\hat{w}_0$ *satisfying the following. For any* $\mathrm{I} \in \mathbf{I}$ *and any* $0 \leqslant p \leqslant k$, *there exists a* $\mathbf{C}$*-linear map*

$$\ell_{\mathrm{I}}^p : \mathrm{H}^0\big(\mathrm{X}, \mathrm{A}^{u\varepsilon}\big) \to \mathscr{O}(\hat{\mathrm{U}}_{\hat{w}_0}),$$

such that for any $(\mathbf{a}, \hat{w}) \in \mathbf{A} \times \hat{\mathrm{U}}_{\hat{w}_0}$, *writing* $\ell_\bullet^p(\mathbf{a}, \hat{w}) = (\ell_{\mathrm{I}}^p(a_{\mathrm{I}})(\hat{w}))_{\mathrm{I}\in\mathbf{I}} \in \mathbf{C}^{\mathbf{I}}$, *one has:*

(1) *The Plücker coordinates of* $\hat{\Phi}(\mathbf{a}, \hat{w})$ *are all vanishing if and only if*

$$\dim \mathrm{Span}\big(\ell_\bullet^0(\mathbf{a}, \hat{w}), \ldots, \ell_\bullet^k(\mathbf{a}, \hat{w})\big) < k+1.$$

(2) *If* $\dim \mathrm{Span}(\ell^0_\bullet(\mathbf{a}, \hat{w}), \dots, \ell^k_\bullet(\mathbf{a}, \hat{w})) = k+1$, *then*

$$\hat{\Phi}(\mathbf{a}, \hat{w}) = \mathrm{Span}\big(\ell^0_\bullet(\mathbf{a}, \hat{w}), \dots, \ell^k_\bullet(\mathbf{a}, \hat{w})\big) \in \mathrm{Gr}_{k+1}\big(\mathbf{C}^{\mathbf{I}}\big).$$

Proof. — From (15), one knows that there exist $\tilde{b}_0, \dots, \tilde{b}_k \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{u\varepsilon})$ such that

$$\hat{\omega}(\tilde{b}_0, \dots, \tilde{b}_k)(\hat{w}_0) \neq 0. \tag{25}$$

Set $w_0 = \nu_k(\hat{w}_0)$ and $x = \pi_{0,k}(w_0)$. Take $\tilde{s} \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^v)$ such that $\tilde{s}(x) \neq 0$, take an open neighborhood $\mathrm{U} \subset (\tilde{s} \neq 0)$ of x and a trivialization of $\mathrm{A}|_{\mathrm{U}}$. Set $s = \tilde{s}^{(r+k)\delta} \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{(r+k)v\delta})$ and define $b_0 = s\tilde{b}_0, \dots, b_k = s\tilde{b}_k \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{u\varepsilon+(r+k)v\delta})$. Moreover, take a neighborhood $\mathrm{U}_{w_0} \subset \mathrm{X}_k$ of w_0 and a family $(\gamma_w)_{w\in \mathrm{U}_{w_0}}$ as in (4). We may assume $\pi_{0,k}(\mathrm{U}_{w_0}) \subset \mathrm{U}$. Take a neighborhood $\hat{\mathrm{U}}_{\hat{w}_0}$ of $\hat{w}_0$ on which $\hat{\omega}(\tilde{b}_0, \dots, \tilde{b}_k)$ never vanishes, and such that $\nu_k(\hat{\mathrm{U}}_{\hat{w}_0}) \subset \mathrm{U}_{w_0}$. For any $m \geqslant 0$, any $\sigma \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^m)$, any $0 \leqslant p \leqslant k$ and any $\hat{w} \in \hat{\mathrm{U}}_{\hat{w}_0}$, define $d^{[p]}_{\mathrm{U}}\sigma(\hat{w}) := d^{[p]}_{\mathrm{U}}\sigma([\gamma_{\nu_k(\hat{w})}]_k)$. This defines an element $d^{[p]}_{\mathrm{U}}\sigma \in \mathscr{O}(\hat{\mathrm{U}}_{\hat{w}_0})$. Similarly, define for each $\mathrm{I} \in \mathbf{I}$ an element $d^{[p]}_{\mathrm{I},\mathrm{U}}\sigma \in \mathscr{O}(\hat{\mathrm{U}}_{\hat{w}_0})$ for any $\sigma \in \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{u\varepsilon})$. Let us fix the trivialization of $\mathscr{O}_{\mathrm{X}_k}(k')|_{\mathrm{U}_{w_0}}$ induced by $(\gamma'_w(0))_{w\in \mathrm{U}_{w_0}} \in \Gamma(\mathrm{U}_{w_0}, \mathscr{O}_{\mathrm{X}_k}(-1))$. Let us also fix a local generator $\mathrm{F}_{\hat{\mathrm{U}}_{\hat{w}_0}} \in \mathscr{O}(\hat{\mathrm{U}}_{\hat{w}_0})$ of the Cartier divisor F and consider, over $\hat{\mathrm{U}}_{\hat{w}_0}$, the induced trivialization for $\nu_k^*\mathscr{O}_{\mathrm{X}_k}(k') \otimes \mathscr{O}_{\hat{\mathrm{X}}_k}(-\mathrm{F})|_{\hat{\mathrm{U}}_{\hat{w}_0}}$.

In this setting, consider the matrix

$$\mathrm{G} := \begin{pmatrix} d^{[0]}_{\mathrm{U}}(b_0) & \cdots & d^{[0]}_{\mathrm{U}}(b_k) \\ \vdots & & \vdots \\ d^{[k]}_{\mathrm{U}}(b_0) & \cdots & d^{[k]}_{\mathrm{U}}(b_k) \end{pmatrix} \in \mathrm{Mat}_{k+1,k+1}\big(\mathscr{O}(\hat{\mathrm{U}}_{\hat{w}_0})\big).$$

Observe that $\det \mathrm{G} = \nu_k^*(\omega(b_0, \dots, b_k)_{\mathrm{U}_{w_0}}) = \mathrm{F}_{\hat{\mathrm{U}}_{\hat{w}_0}} \hat{\omega}(b_0, \dots, b_k)_{\hat{\mathrm{U}}_{\hat{w}_0}}$ is not identically zero because from (25) and (8) it follows that $\hat{\omega}(b_0, \dots, b_k)$ never vanishes on $\hat{\mathrm{U}}_{\hat{w}_0}$. Let us now prove that one can define, for any $\mathrm{I} \in \mathbf{I}$, linear maps $\ell^0_{\mathrm{I}}, \dots, \ell^k_{\mathrm{I}} : \mathrm{H}^0(\mathrm{X}, \mathrm{A}^{u\varepsilon}) \to \mathscr{O}(\hat{\mathrm{U}}_{\hat{w}_0})$ by

$$\begin{pmatrix} \ell^0_{\mathrm{I}}(a_{\mathrm{I}}) \\ \vdots \\ \ell^k_{\mathrm{I}}(a_{\mathrm{I}}) \end{pmatrix} = \mathrm{G}^{-1} \begin{pmatrix} d^{[0]}_{\mathrm{I},\mathrm{U}}(a_{\mathrm{I}}) \\ \vdots \\ d^{[k]}_{\mathrm{I},\mathrm{U}}(a_{\mathrm{I}}) \end{pmatrix} = \frac{1}{\tau^{r\mathrm{I}}_{\mathrm{U}}}\mathrm{G}^{-1} \begin{pmatrix} d^{[0]}_{\mathrm{U}}(a_{\mathrm{I}}\tau^{(r+k)\mathrm{I}}) \\ \vdots \\ d^{[k]}_{\mathrm{U}}(a_{\mathrm{I}}\tau^{(r+k)\mathrm{I}}) \end{pmatrix} \in \mathrm{Mat}_{k+1,1}\big(\mathscr{O}(\hat{\mathrm{U}}_{\hat{w}_0})\big). \tag{26}$$

The key point is to see that this is well defined, namely that for any $0 \leqslant p \leqslant k$, we have $\ell^p_{\mathrm{I}}(a_{\mathrm{I}}) \in \mathscr{O}(\hat{\mathrm{U}}_{\hat{w}_0})$. By multi-linearity of the determinant and in view of Lemma 3.3 it follows that there exists

$$\omega_{p,\mathrm{I}}(b_0, \dots, b_{p-1}, a_{\mathrm{I}}, b_{p+1}, \dots, b_k) \in \mathrm{H}^0\big(\mathrm{X}_k, \mathscr{O}_{\mathrm{X}_k}\big(k'\big) \otimes \mathrm{A}^{(k+1)(u\varepsilon+(r+k)v\delta)-vr\delta}\big)$$

such that

$$\omega\big(b_0, \dots, b_{p-1}, a_{\mathrm{I}}\tau^{(r+k)\mathrm{I}}, b_{p+1}, \dots, b_k\big) \\ = \tau^{r\mathrm{I}}\omega_{p,\mathrm{I}}(b_0, \dots, b_{p-1}, a_{\mathrm{I}}, b_{p+1}, \dots, b_k).$$

Lemma 3.5 then implies that $\omega_{p,\mathrm{I}}(b_0, \dots, a_{\mathrm{I}}, \dots, b_k)$ vanishes along $\mathfrak{w}_\infty(\mathrm{X}_k)$, and therefore, there exists $\hat{\omega}_{p,\mathrm{I}}(b_0, \dots, a_{\mathrm{I}}, \dots, b_k) \in \mathrm{H}^0(\hat{\mathrm{X}}_k, \nu_k^*(\mathscr{O}_{\mathrm{X}_k}(k') \otimes \pi_{0,k}^*\mathrm{A}^{(k+1)(u\varepsilon+(r+k)v\delta)-vr\delta}) \otimes \mathscr{O}_{\hat{\mathrm{X}}_k}(-\mathrm{F}))$ such that

$$\nu_k^*\omega_{p,\mathrm{I}}(b_0, \dots, a_{\mathrm{I}}, \dots, b_k) = \mathrm{F}\cdot\hat{\omega}_{p,\mathrm{I}}(b_0, \dots, a_{\mathrm{I}}, \dots, b_k).$$

For each $0 \leqslant p \leqslant k$, applying Cramer's rule, one obtains from the definition of ω and $\hat{\omega}$ that

$$\begin{aligned} \ell_{\mathrm{I}}^p(a_{\mathrm{I}}) &= \frac{1}{\tau_{\mathrm{U}}^{r\mathrm{I}}\det\mathrm{G}} \\ &\quad\times\begin{vmatrix} d_{\mathrm{U}}^{[0]}(b_0) & \cdots & d_{\mathrm{U}}^{[0]}(b_{p-1}) & d_{\mathrm{U}}^{[0]}(a_{\mathrm{I}}\tau^{(r+k)\mathrm{I}}) & d_{\mathrm{U}}^{[0]}(b_{p+1}) & \cdots & d_{\mathrm{U}}^{[0]}(b_k) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ d_{\mathrm{U}}^{[k]}(b_0) & \cdots & d_{\mathrm{U}}^{[k]}(b_{p-1}) & d_{\mathrm{U}}^{[k]}(a_{\mathrm{I}}\tau^{(r+k)\mathrm{I}}) & d_{\mathrm{U}}^{[k]}(b_{p+1}) & \cdots & d_{\mathrm{U}}^{[k]}(b_k) \end{vmatrix} \\ &= \frac{\nu_k^*\omega(b_0, \dots, b_{p-1}, a_{\mathrm{I}}\tau^{(r+k)\mathrm{I}}, b_{p+1}, \dots, b_k)_{\mathrm{U}_{w_0}}}{\tau_{\mathrm{U}}^{r\mathrm{I}}\nu_k^*\omega(b_0, \dots, b_k)_{\mathrm{U}_{w_0}}} \\ &= \frac{\nu_k^*\omega_{p,\mathrm{I}}(b_0, \dots, b_{p-1}, a_{\mathrm{I}}, b_{p+1}, \dots, b_k)_{\mathrm{U}_{w_0}}}{\nu_k^*\omega(b_0, \dots, b_k)_{\mathrm{U}_{w_0}}} \\ &= \frac{\mathrm{F}_{\hat{\mathrm{U}}_{\hat{w}_0}}\hat{\omega}_{p,\mathrm{I}}(b_0, \dots, a_{\mathrm{I}}, \dots, b_k)_{\hat{\mathrm{U}}_{\hat{w}_0}}}{\mathrm{F}_{\hat{\mathrm{U}}_{\hat{w}_0}}\hat{\omega}(b_0, \dots, b_k)_{\hat{\mathrm{U}}_{\hat{w}_0}}} = \frac{\hat{\omega}_{p,\mathrm{I}}(b_0, \dots, a_{\mathrm{I}}, \dots, b_k)_{\hat{\mathrm{U}}_{\hat{w}_0}}}{\hat{\omega}(b_0, \dots, b_k)_{\hat{\mathrm{U}}_{\hat{w}_0}}}, \end{aligned}$$

where we used (14). The desired holomorphicity follows because $\hat{\omega}(b_0, \dots, b_k)$ never vanishes on $\hat{\mathrm{U}}_{\hat{w}_0}$.

With the notation of the statement of the lemma, a straightforward computation shows that the Plücker coordinates of $\mathrm{Span}(\ell_\bullet^0(\mathbf{a}, \hat{w}), \dots, \ell_\bullet^k(\mathbf{a}, \hat{w}))$ are given by

$$\big(\hat{\omega}_{\mathrm{I}_0,\dots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k})(\hat{w})\big)_{\mathrm{I}_0,\dots,\mathrm{I}_k\in\mathbf{I}} \in \Lambda^{k+1}\mathbf{C}^{\mathbf{I}} \mod \mathbf{C}^*. \tag{27}$$

Indeed, for any $\mathrm{I}_0, \dots, \mathrm{I}_k \in \mathbf{I}$,

$$\begin{aligned} \begin{vmatrix} \ell_{\mathrm{I}_0}^0(a_{\mathrm{I}_0}) & \cdots & \ell_{\mathrm{I}_k}^0(a_{\mathrm{I}_k}) \\ \vdots & & \vdots \\ \ell_{\mathrm{I}_0}^k(a_{\mathrm{I}_0}) & \cdots & \ell_{\mathrm{I}_k}^k(a_{\mathrm{I}_k}) \end{vmatrix}(\hat{w}) &= \frac{1}{\det\mathrm{G}}\begin{vmatrix} d_{\mathrm{I}_0,\mathrm{U}}^{[0]}(a_{\mathrm{I}_0}) & \cdots & d_{\mathrm{I}_k,\mathrm{U}}^{[0]}(a_{\mathrm{I}_k}) \\ \vdots & & \vdots \\ d_{\mathrm{I}_0,\mathrm{U}}^{[k]}(a_{\mathrm{I}_0}) & \cdots & d_{\mathrm{I}_k,\mathrm{U}}^{[k]}(a_{\mathrm{I}_k}) \end{vmatrix}(\hat{w}) \\ &= \frac{\nu_k^*\omega_{\mathrm{I}_0,\dots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k})_{\mathrm{U}_{w_0}}(\hat{w})}{\nu_k^*\omega(b_0, \dots, b_k)_{\mathrm{U}_{w_0}}(\hat{w})} \end{aligned}$$

$$= \frac{\hat{\omega}_{\mathrm{I}_0,\dots,\mathrm{I}_k}(a_{\mathrm{I}_0},\dots,a_{\mathrm{I}_k})_{\hat{\mathrm{U}}_{\hat{w}_0}}(\hat{w})}{\hat{\omega}(b_0,\dots,b_k)_{\hat{\mathrm{U}}_{\hat{w}_0}}(\hat{w})}.$$

Since $\hat{\omega}_{\hat{\mathrm{U}}_k}(b_0,\dots,b_k)(\hat{w})$ is independent of $\mathrm{I}_0,\dots,\mathrm{I}_k$, this proves (27), which concludes the proof. □

Before continuing, we need to introduce some notation. For any $x\in\mathrm{X}$, define

$$\mathrm{N}_x := \#\big\{j\in\{0,\dots,\mathrm{N}\}\ \big|\ \tau_j(x)\neq 0\big\} \quad \text{and} \quad \mathbf{I}_x := \big\{\mathrm{I}\in\mathbf{I}\ \big|\ \tau^{\mathrm{I}}(x)\neq 0\big\}.$$

Observe that since the τ_j's are in general position, and since $\mathrm{N}\geqslant n$, one has $\mathrm{N}_x\geqslant 1$ for all $x\in\mathrm{X}$. Let us also define

$$\Sigma := \{x\in\mathrm{X}\ |\ \mathrm{N}_x = 1\} \quad \text{and} \quad \mathrm{X}^\circ := \mathrm{X}\setminus\Sigma = \{x\in\mathrm{X}\ |\ \mathrm{N}_x\geqslant 2\}.$$

If $\mathrm{N}>n$ then $\mathrm{N}_x\geqslant 2$ for all $x\in\mathrm{X}$, therefore $\Sigma=\varnothing$ and $\mathrm{X}^\circ=\mathrm{X}$. If $\mathrm{N}=n$ then $\dim\Sigma = 0$. Observe moreover that

$$\#\mathbf{I}_x = \binom{\mathrm{N}_x-1+\delta}{\delta} \quad \text{for all } x\in\mathrm{X} \quad \text{and therefore} \quad \#\mathbf{I}_x\geqslant\delta+1 \quad \text{for all } x\in\mathrm{X}^\circ. \tag{28}$$

For any $x\in\mathrm{X}$, write $\mathbf{C}^{\mathbf{I}_x}=\bigoplus_{\mathrm{I}\in\mathbf{I}_x}\mathbf{C}$. One obtains a natural projection map $\rho_x : (\mathbf{C}^{\mathbf{I}})^{k+1}\to(\mathbf{C}^{\mathbf{I}_x})^{k+1}$.

We will from now on suppose that $\varepsilon\geqslant m_\infty$. With the notation of Lemma 3.6, it is natural to consider, given $\hat{w}_0\in\hat{\mathrm{X}}_k$, the map

$$\begin{aligned} \hat{\varphi}_{\hat{w}_0} : \mathbf{A} &\to \big(\mathbf{C}^{\mathbf{I}}\big)^{k+1} \\ \mathbf{a} &\mapsto \big(\ell^0_\bullet(\mathbf{a},\hat{w}_0),\dots,\ell^k_\bullet(\mathbf{a},\hat{w}_0)\big). \end{aligned} \tag{29}$$

This map is not canonical since it depends on the choices made during the proof of Lemma 3.6. Nevertheless, in view of this lemma, we can use it to obtain crucial information on $\hat{\Phi}(\bullet,\hat{w}_0)$. The map $\hat{\varphi}_{\hat{w}_0}$ is particularly interesting because it is linear, hence much simpler to study than $\hat{\Phi}(\bullet,\hat{w}_0)$. We will need precise information on the rank of $\hat{\varphi}_{\hat{w}_0}$.

Lemma **3.7.** — *Same notation as above. For $x=\pi_{0,k}\circ\nu_k(\hat{w}_0)$, one has*

$$\operatorname{rank}\rho_x\circ\hat{\varphi}_{\hat{w}_0} = (k+1)\#\mathbf{I}_x. \tag{30}$$

Proof. — Take the notation of the proof of Lemma 3.6. Up to the isomorphism $(\mathbf{C}^{\mathbf{I}_x})^{k+1}\cong(\mathbf{C}^{k+1})^{\mathbf{I}_x}$, one can see $\rho_x\circ\hat{\varphi}_{\hat{w}_0}$ as the map

$$\rho_x\circ\hat{\varphi}_{\hat{w}_0} = (\hat{\varphi}_{\mathrm{I}})_{\mathrm{I}\in\mathbf{I}_x},$$

where for each $\mathrm{I} \in \mathbf{I}_x$, $\hat{\varphi}_{\mathrm{I}}$ is defined by

$$\hat{\varphi}_{\mathrm{I}} : \mathrm{H}^0\big(\mathrm{X}, \mathrm{A}^{u\varepsilon}\big) \to \mathbf{C}^{k+1}$$
$$a_{\mathrm{I}} \mapsto \big(\ell^0_{\mathrm{I}}(a_{\mathrm{I}})(\hat{w}_0), \dots, \ell^k_{\mathrm{I}}(a_{\mathrm{I}})(\hat{w}_0)\big).$$

Observe that $\operatorname{rank}(\rho_x \circ \hat{\varphi}_{\hat{w}_0}) = \sum_{\mathrm{I}\in\mathbf{I}_x} \operatorname{rank} \hat{\varphi}_{\mathrm{I}}$. Therefore, to prove (30), it suffices to show that for any $\mathrm{I} \in \mathbf{I}_x$, one has

(**31**) $$\operatorname{rank} \hat{\varphi}_{\mathrm{I}} = k + 1.$$

To do so, consider the family $(\tilde{b}_0, \dots, \tilde{b}_k)$ as in (25) above. Observe that (26) infers that

$$\begin{aligned}
\begin{vmatrix} \ell^0_{\mathrm{I}}(\tilde{b}_0) & \cdots & \ell^0_{\mathrm{I}}(\tilde{b}_k) \\ \vdots & & \vdots \\ \ell^k_{\mathrm{I}}(\tilde{b}_0) & \cdots & \ell^k_{\mathrm{I}}(\tilde{b}_k) \end{vmatrix}
&= \frac{1}{\tau_{\mathrm{U}}^{r(k+1)\mathrm{I}} \det \mathrm{G}} \begin{vmatrix} d^{[0]}_{\mathrm{U}}(\tilde{b}_0 \tau^{(r+k)\mathrm{I}}) & \cdots & d^{[0]}_{\mathrm{U}}(\tilde{b}_k \tau^{(r+k)\mathrm{I}}) \\ \vdots & & \vdots \\ d^{[k]}_{\mathrm{U}}(\tilde{b}_0 \tau^{(r+k)\mathrm{I}}) & \cdots & d^{[k]}_{\mathrm{U}}(\tilde{b}_k \tau^{(r+k)\mathrm{I}}) \end{vmatrix} \\
&= \frac{\nu_k^* \omega(\tilde{b}_0 \tau^{(r+k)\mathrm{I}}, \dots, \tilde{b}_k \tau^{(r+k)\mathrm{I}})_{\mathrm{U}_{w_0}}}{\tau_{\mathrm{U}}^{r(k+1)\mathrm{I}} \nu_k^* \omega(b_0, \dots, b_k)_{\mathrm{U}_{w_0}}} \\
&= \frac{\tau_{\mathrm{U}}^{(k+1)(r+k)\mathrm{I}} \nu_k^* \omega(\tilde{b}_0, \dots, \tilde{b}_k)_{\mathrm{U}_{w_0}}}{s_{\mathrm{U}}^{k+1} \tau_{\mathrm{U}}^{r(k+1)\mathrm{I}} \nu_k^* \omega(\tilde{b}_0, \dots, \tilde{b}_k)_{\mathrm{U}_{w_0}}} \\
&= \frac{\tau_{\mathrm{U}}^{k(k+1)\mathrm{I}}}{s_{\mathrm{U}}^{k+1}}
\end{aligned}$$

(recall that $s(x) \neq 0$). Since we supposed that $\tau^{\mathrm{I}}(x) \neq 0$, this determinant is non-zero when evaluated at the point $\hat{w}_0$. This implies that $\hat{\varphi}_{\mathrm{I}}(\tilde{b}_0) \wedge \cdots \wedge \hat{\varphi}_{\mathrm{I}}(\tilde{b}_k) \neq 0$. Hence $\operatorname{rank} \hat{\varphi}_{\mathrm{I}} = k + 1$, which proves (30). □

From this we will be able to control the indeterminacy locus of $\hat{\Phi}$. Let us define $\hat{\mathrm{X}}^\circ_k := (\pi_{0,k} \circ \nu_k)^{-1}(\mathrm{X}^\circ)$.

Proposition **3.8.** — *Suppose* $\mathrm{N} \geqslant n \geqslant 2$, $k \geqslant 1$, $\varepsilon \geqslant m_\infty$ *and* $\delta \geqslant n(k+1)$. *Then there exists a non-empty Zariski open subset* $\mathbf{A}_{\mathrm{def}} \subset \mathbf{A}_{\mathrm{sm}}$ *such that* $\hat{\Phi}|_{\mathbf{A}_{\mathrm{def}} \times \hat{\mathrm{X}}^\circ_k}$ *is a (regular) morphism.*

Proof. — The indeterminacy locus of $\hat{\Phi}|_{\mathbf{A} \times \hat{\mathrm{X}}^\circ_k}$ is contained in

$$\mathrm{Z} = \big\{(\mathbf{a}, \hat{w}) \in \mathbf{A} \times \hat{\mathrm{X}}^\circ_k \ \big|\ \hat{\omega}_{\mathrm{I}_0,\dots,\mathrm{I}_k}(a_{\mathrm{I}_0}, \dots, a_{\mathrm{I}_k})(\hat{w}) = 0 \ \forall \mathrm{I}_0, \dots, \mathrm{I}_k \in \mathbf{I}\big\}.$$

Denoting by $\hat{\mathrm{pr}}_1 : \mathbf{A} \times \hat{\mathrm{X}}^\circ_k \to \mathbf{A}$ and $\hat{\mathrm{pr}}_2 : \mathbf{A} \times \hat{\mathrm{X}}^\circ_k \to \hat{\mathrm{X}}^\circ_k$ the two natural projections, we aim to prove that Z does not dominate $\mathbf{A}$ via $\hat{\mathrm{pr}}_1$. It is sufficient to prove that

(**32**) $$\dim \mathrm{Z} < \dim \mathbf{A}.$$

Fix $\hat{w}_0 \in \hat{X}_k^\circ$, set $x = \pi_{0,k} \circ \nu_k(\hat{w}_0)$ and define $Z_{\hat{w}_0} := Z \cap \hat{\mathrm{pr}}_2^{-1}(\hat{w}_0)$. Consider the map $\hat{\varphi}_{\hat{w}_0}$ defined by (29). From Lemma 3.6, one sees that $\hat{\mathrm{pr}}_1(Z_{\hat{w}_0}) = \hat{\varphi}_{\hat{w}_0}^{-1}(\Delta)$, where

$$\Delta := \left\{ \left(v_\bullet^0, \dots, v_\bullet^k\right) \in \left(\mathbf{C}^{\mathbf{I}}\right)^{k+1} \mid \dim \mathrm{Span}\left(v_\bullet^0, \dots, v_\bullet^k\right) < k+1 \right\}.$$

Defining

$$\Delta_x := \left\{ \left(v_\bullet^0, \dots, v_\bullet^k\right) \in \left(\mathbf{C}^{\mathbf{I}_x}\right)^{k+1} \mid \dim \mathrm{Span}\left(v_\bullet^0, \dots, v_\bullet^k\right) < k+1 \right\},$$

we certainly have $\Delta \subset \rho_x^{-1}(\Delta_x)$, and therefore

$$\hat{\mathrm{pr}}_1(Z_{\hat{w}_0}) \subset (\rho_x \circ \hat{\varphi}_{\hat{w}_0})^{-1}(\Delta_x).$$

Observe that $\dim \Delta_x = k\#\mathbf{I}_x + k$. Moreover, one has $\mathrm{rank}(\rho_x \circ \hat{\varphi}_{\hat{w}_0}) = (k+1)\#\mathbf{I}_x$ in view of Lemma 3.7. Therefore

$$\begin{aligned} \dim Z_{\hat{w}_0} = \dim \hat{\mathrm{pr}}_1(Z_{\hat{w}_0}) &\leqslant \dim(\rho_x \circ \hat{\varphi}_{\hat{w}_0})^{-1}(\Delta_x) \\ &\leqslant \dim \Delta_x + \dim \ker(\rho_x \circ \hat{\varphi}_{\hat{w}_0}) \\ &\leqslant k\#\mathbf{I}_x + k + \dim \mathbf{A} - (k+1)\#\mathbf{I}_x = \dim \mathbf{A} + k - \#\mathbf{I}_x. \end{aligned}$$

Hence,

$$\begin{aligned} \dim Z &\leqslant \dim \hat{X}_k + \dim \mathbf{A} + k - \min_{x \in X^\circ}(\#\mathbf{I}_x) \\ &\leqslant n + k(n-1) + \dim \mathbf{A} + k - \delta - 1 < \dim \mathbf{A} \end{aligned} \tag{33}$$

in view of (28) and of our hypothesis on δ. It now suffices to take $\mathbf{A}_{\mathrm{def}} := (\mathbf{A} \setminus \overline{\hat{\mathrm{pr}}_1(Z)}) \cap \mathbf{A}_{\mathrm{sm}}$. □

Remark **3.9.** — The hypothesis on δ in Proposition 3.8, while sufficient for our purposes, is not optimal. It follows immediately from (33) and (28) that the same conclusion would still hold if δ satisfies for instance

$$\binom{N - n + \delta}{\delta} > \dim X_k + k. \tag{34}$$

3.3. *Maps to families of negative dimensional complete intersection varieties.* — Suppose from now on that $N \geqslant n \geqslant 2$, $k \geqslant N-1$, $\varepsilon \geqslant m_\infty$ and that $\delta \geqslant n(k+1)$ (or that δ satisfies (34)). To complete the set-up for our proof we need one more ingredient: to construct suitable maps to families of "negative dimensional complete intersection varieties". To do this properly, we need to consider the natural stratification on X induced by the vanishing of the τ_j's. The necessity of using this stratification comes from our particular choice of equation $F(\mathbf{a})$, and seems unavoidable. It was already present less explicitly in [6], then it was developed and used in a systematic way in [63], and was also crucial in [7] and [64].

For any $\mathrm{J} \subset \{0, \dots, \mathrm{N}\}$, define

$$\mathrm{X}_{\mathrm{J}} := \big\{x \in \mathrm{X} \mid \tau_j(x) = 0 \Leftrightarrow j \in \mathrm{J}\big\},$$
$$\mathbf{I}_{\mathrm{J}} := \big\{\mathrm{I} \in \mathbf{I} \mid \mathrm{Supp}(\mathrm{I}) \subset \{0, \dots, \mathrm{N}\} \setminus \mathrm{J}\big\}.$$

Observe that $x \in \mathrm{X}_{\mathrm{J}}$ if and only if $\mathbf{I}_x = \mathbf{I}_{\mathrm{J}}$. Since the τ_j's are in general position, one obtains that

$$\dim \mathrm{X}_{\mathrm{J}} = \max\{-1, n - \#\mathrm{J}\},$$

where by $\dim \mathrm{X}_{\mathrm{J}} = -1$ we mean $\mathrm{X}_{\mathrm{J}} = \varnothing$. Therefore, $(\mathrm{X}_{\mathrm{J}})_{\#\mathrm{J} \leqslant n}$ defines a stratification on X. For any $\mathrm{J} \subset \{0, \dots, \mathrm{N}\}$, let us define

$$\mathbf{P}_{\mathrm{J}} := \big\{[\mathrm{T}_0, \dots, \mathrm{T}_{\mathrm{N}}] \in \mathbf{P}^{\mathrm{N}} \mid \mathrm{T}_j = 0 \ \text{ if } j \in \mathrm{J}\big\}.$$

One can naturally identify $\mathbf{C}^{\mathbf{I}_{\mathrm{J}}} := \bigoplus_{\mathrm{I} \in \mathbf{I}_{\mathrm{J}}} \mathbf{C}$ with $\mathrm{H}^0(\mathbf{P}_{\mathrm{J}}, \mathscr{O}_{\mathbf{P}_{\mathrm{J}}}(\delta)) \cong \mathbf{C}[(\mathrm{T}_{j'})_{j' \in \{0,\dots,\mathrm{N}\} \setminus \mathrm{J}}]_{\delta}$, the space of homogenous degree δ polynomials in the variables $\mathrm{T}_{j'}$ with $j' \notin \mathrm{J}$. This identification is realized by the map

$$(c_{\mathrm{I}})_{\mathrm{I} \in \mathbf{I}_{\mathrm{J}}} \mapsto \sum_{\mathrm{I} \in \mathbf{I}_{\mathrm{J}}} c_{\mathrm{I}} \mathrm{T}^{\mathrm{I}}.$$

For $\mathrm{J} = \varnothing$ this just gives the natural identification between $\mathbf{C}^{\mathbf{I}}$ and $\mathrm{H}^0(\mathbf{P}^{\mathrm{N}}, \mathscr{O}_{\mathbf{P}^{\mathrm{N}}}(\delta)) \cong \mathbf{C}[\mathrm{T}_0, \dots, \mathrm{T}_{\mathrm{N}}]_{\delta}$. Given $\Delta \in \mathrm{Gr}_{k+1}(\mathbf{C}^{\mathbf{I}}) \cong \mathrm{Gr}_{k+1}(\mathbf{C}[\mathrm{T}_0, \dots, \mathrm{T}_{\mathrm{N}}]_{\delta})$ and $[\mathrm{T}] \in \mathbf{P}^{\mathrm{N}}$, write $\Delta([\mathrm{T}]) = 0$ if $\mathrm{P}(\mathrm{T}) = 0$ for all $\mathrm{P} \in \Delta \subset \mathbf{C}[\mathrm{T}_0, \dots, \mathrm{T}_{\mathrm{N}}]_{\delta}$. If $\Delta = \mathrm{Span}(\mathrm{P}_0, \dots, \mathrm{P}_k)$, this condition is equivalent to

$$\mathrm{P}_0(\mathrm{T}) = 0, \dots, \mathrm{P}_k(\mathrm{T}) = 0. \tag{35}$$

Let us define the family

$$\mathscr{Y} := \big\{\big(\Delta, [\mathrm{T}]\big) \in \mathrm{Gr}_{k+1}\big(\mathbf{C}^{\mathbf{I}}\big) \times \mathbf{P}^{\mathrm{N}} \mid \Delta\big([\mathrm{T}]\big) = 0\big\}.$$

Observe that $\mathscr{Y}$ is smooth. Indeed, if we denote by S the vector bundle on $\mathbf{P}^n$ defined by the short exact sequence

$$0 \to \mathrm{S} \to \mathrm{H}^0\big(\mathbf{P}^{\mathrm{N}}, \mathscr{O}_{\mathbf{P}^{\mathrm{N}}}(\delta)\big) \otimes \mathscr{O}_{\mathbf{P}^{\mathrm{N}}} \stackrel{\mathrm{ev}}{\to} \mathscr{O}_{\mathbf{P}^{\mathrm{N}}}(\delta) \to 0,$$

then $\mathscr{Y}$ is isomorphic to the Grassmann bundle $\mathrm{Gr}_{k+1}(\mathrm{S}) \to \mathbf{P}^{\mathrm{N}}$. Consider the map

$$\begin{aligned} \hat{\Psi} : \mathbf{A}_{\mathrm{def}} \times \hat{\mathrm{X}}_k^{\circ} &\to \mathrm{Gr}_{k+1}\big(\mathbf{C}^{\mathbf{I}}\big) \times \mathbf{P}^{\mathrm{N}} \\ (\mathbf{a}, \hat{w}) &\mapsto \big(\hat{\Phi}(\mathbf{a}, \hat{w}), \big[\tau^r(\hat{w})\big]\big), \end{aligned}$$

where $[\tau^r(\hat{w})] := [\tau_0^r(\pi_{0,k} \circ \nu_k(\hat{w})), \dots, \tau_{\mathrm{N}}^r(\pi_{0,k} \circ \nu_k(\hat{w}))]$. Recall from Section 3.1 the definition of $\mathscr{H} \subset \mathbf{A}_{\mathrm{sm}} \times \mathrm{X}$ and $\hat{\mathscr{H}}_k^{\mathrm{rel}} \subset \mathbf{A}_{\mathrm{sm}} \times \hat{\mathrm{X}}_k$. We will be interested in $\hat{\Psi}|_{\hat{\mathscr{H}}_k^{\mathrm{rel}}}$ and for this reason we will restrict ourselves to the locus where this map is regular. Let us therefore define

$$\mathbf{A}^{\circ}_{\mathrm{def}} := \mathbf{A}_{\mathrm{def}} \cap \{\mathbf{a} \in \mathbf{A} \mid \mathrm{H}_{\mathbf{a}} \cap \Sigma = \varnothing\}.$$

Since Σ is at most a finite number of points, $\mathbf{A}^{\circ}_{\mathrm{def}}$ is a non-empty Zariski open subset of $\mathbf{A}$. Moreover, it follows from Proposition 3.8 that $\hat{\Psi}|_{\hat{\mathscr{H}}^{\mathrm{rel}}_k \cap (\mathbf{A}^{\circ}_{\mathrm{def}} \times \hat{\mathrm{X}}_k)}$ is regular since $\hat{\mathscr{H}}^{\mathrm{rel}}_k \cap (\mathbf{A}^{\circ}_{\mathrm{def}} \times \hat{\mathrm{X}}_k) \subset \mathbf{A}_{\mathrm{def}} \times \hat{\mathrm{X}}^{\circ}_k$.

For any $\mathrm{J} \subset \{0, \dots, \mathrm{N}\}$, set

$$\mathscr{Y}_{\mathrm{J}} := \mathscr{Y} \cap \left(\mathrm{Gr}_{k+1}\left(\mathbf{C}^{\mathbf{I}}\right) \times \mathbf{P}_{\mathrm{J}}\right) \subset \mathrm{Gr}_{k+1}\left(\mathbf{C}^{\mathbf{I}}\right) \times \mathbf{P}^{\mathrm{N}}.$$

Observe that $\mathscr{Y}_{\mathrm{J}}$ is smooth since $\mathscr{Y}_{\mathrm{J}}$ is isomorphic to the Grassmann bundle $\mathrm{Gr}_{k+1}(\mathrm{S}|_{\mathbf{P}_{\mathrm{J}}}) \to \mathbf{P}_{\mathrm{J}}$. Set also $\hat{\mathrm{X}}_{k,\mathrm{J}} := \nu_k^{-1}(\pi_{0,k}^{-1}(\mathrm{X}_{\mathrm{J}}))$, and let us define

$$\hat{\mathscr{H}}^{\mathrm{rel}}_{k,\mathrm{J}} := \hat{\mathscr{H}}^{\mathrm{rel}}_k \cap \left(\mathbf{A}^{\circ}_{\mathrm{def}} \times \hat{\mathrm{X}}_{k,\mathrm{J}}\right) \subset \hat{\mathscr{H}}^{\mathrm{rel}}_k \cap \left(\mathbf{A}^{\circ}_{\mathrm{def}} \times \hat{\mathrm{X}}_k\right).$$

One has the following.

Proposition **3.10.** — *For any* $\mathrm{J} \subset \{0, \dots, \mathrm{N}\}$, *when restricted to* $\hat{\mathscr{H}}^{\mathrm{rel}}_{k,\mathrm{J}}$, *the morphism* $\hat{\Psi}$ *factors through* $\mathscr{Y}_{\mathrm{J}}$*:*

$$\hat{\Psi}|_{\hat{\mathscr{H}}^{\mathrm{rel}}_{k,\mathrm{J}}} : \hat{\mathscr{H}}^{\mathrm{rel}}_{k,\mathrm{J}} \to \mathscr{Y}_{\mathrm{J}} \subset \mathrm{Gr}_{k+1}\left(\mathbf{C}^{\mathbf{I}}\right) \times \mathbf{P}^{\mathrm{N}}.$$

Proof. — It suffices to prove that $\hat{\Psi}$ restricted to $\mathbf{A}_{\mathrm{def}} \times \hat{\mathrm{X}}^{\circ}_{k,\mathrm{J}}$ factors through $\mathrm{Gr}_{k+1}(\mathbf{C}^{\mathbf{I}}) \times \mathbf{P}_{\mathrm{J}}$ and that $\hat{\Psi}$ restricted to $\hat{\mathscr{H}}^{\mathrm{rel}}_k$ factors through $\mathscr{Y}$. The first statement is straightforward to prove. Therefore we now focus on the second one. Since $\hat{\Phi} = \Phi \circ \nu_k$, one sees that it suffices to prove that the rational map

$$\begin{aligned} \Psi : \mathbf{A} \times \mathrm{X}_k &\dashrightarrow \mathrm{Gr}_{k+1}\left(\mathbf{C}^{\mathbf{I}}\right) \times \mathbf{P}^{\mathrm{N}} \\ (\mathbf{a}, w) &\mapsto \left(\Phi(\mathbf{a}, w), \left[\tau^r(w)\right]\right) \end{aligned}$$

factors through $\mathscr{Y}$ when restricted to $\mathscr{H}^{\mathrm{rel}}_k \subset \mathbf{A}_{\mathrm{sm}} \times \mathrm{X}_k$. Fix $(\mathbf{a}, w_0) \in \mathscr{H}^{\mathrm{rel}}_k$ outside the indeterminacy locus of Φ. Take a neighborhood U_{w_0} of w_0 in X_k, a family $(\gamma_w)_{w \in \mathrm{U}_{w_0}}$ as in (4) and a neighborhood U of $\pi_{0,k}(w_0)$ as in Lemma 3.3. By construction, $\mathrm{H}_{k,\mathbf{a}} := (\rho \circ \pi_{0,k})^{-1}(\mathbf{a})$ equals the kth order jet space $\mathrm{H}_{\mathbf{a},k} \subset \mathrm{X}_k$ associated to $\mathrm{H}_{\mathbf{a}} \subset \mathrm{X}$. Therefore $[\gamma_{w_0}]_k \in \mathrm{J}_k\mathrm{H}_{\mathbf{a}} \cap p_k^{-1}(\mathrm{U})$, which implies that $d^{[p]}_{\mathrm{U}}\mathrm{F}(\mathbf{a})([\gamma_{w_0}]_k) = 0$ for all $0 \leqslant p \leqslant k$. But by Lemma 3.3,

$$d^{[p]}_{\mathrm{U}}\mathrm{F}(\mathbf{a}) = \sum_{\mathrm{I} \in \mathbf{I}} d^{[p]}_{\mathrm{U}}\left(a_{\mathrm{I}}\tau^{(r+k)\mathrm{I}}\right) = \sum_{\mathrm{I} \in \mathbf{I}} \left(d^{[p]}_{\mathrm{I},\mathrm{U}}(a_{\mathrm{I}})\right)\tau^{r\mathrm{I}}.$$

It then follows from the definition of $\Phi(\mathbf{a}, w_0)$, the definition of $\mathscr{Y}$, (23) and (35), that $\Phi(\mathbf{a}, w_0) \in \mathscr{Y}$. □

As in [7], the key argument in the proof of Theorem 3.2 relies on the study of the non-finite locus of the families $\mathscr{Y}_J$. For $J \subset \{0, \ldots, N\}$, denote by $p_J : \mathscr{Y}_J \to \mathrm{Gr}_{k+1}(\mathbf{C}^{\mathbf{I}})$ the first projection, and define

$$\mathrm{E}_J := \left\{ y \in \mathscr{Y} \mid \dim_y\left(p_J^{-1}\left(p_J(y)\right)\right) > 0 \right\}$$
$$\mathbf{G}_J^{\infty} := p_J(\mathrm{E}_J) \subset \mathrm{Gr}_{k+1}\left(\mathbf{C}^{\mathbf{I}}\right).$$

The next lemma will be crucial. Let us denote, for any $J \subset \{0, \ldots, N\}$, $\hat{X}^{\circ}_{k,J} := \hat{X}_{k,J} \cap \hat{X}^{\circ}_k$.

Lemma **3.11.** — *Fix* $J \subset \{0, \ldots, N\}$. *If* $\delta \geqslant \dim \hat{X}_k$, *then there exists a non-empty Zariski open subset* $\mathbf{A}_J \subset \mathbf{A}_{\mathrm{def}}$, *such that*

$$\hat{\Phi}^{-1}\left(\mathbf{G}_J^{\infty}\right) \cap \left(\mathbf{A}_J \times \hat{X}^{\circ}_{k,J}\right) = \varnothing. \tag{36}$$

Proof. — For $J \subset \{0, \ldots, N\}$, define the following analogues of $\mathscr{Y}_J$ parametrized by affine spaces.

$$\widetilde{\mathscr{Y}}_{1,J} := \left\{ \left(P_0, \ldots, P_k, [T]\right) \in \left(\mathbf{C}^{\mathbf{I}}\right)^{k+1} \times \mathbf{P}_J \mid P_0(T) = 0, \ldots, P_k(T) = 0 \right\},$$
$$\widetilde{\mathscr{Y}}_{2,J} := \left\{ \left(P_0, \ldots, P_k, [T]\right) \in \left(\mathbf{C}^{\mathbf{I}_J}\right)^{k+1} \times \mathbf{P}_J \mid P_0(T) = 0, \ldots, P_k(T) = 0 \right\}.$$

Here we used the identifications $\mathbf{C}^{\mathbf{I}} \cong H^0(\mathbf{P}^N, \mathscr{O}_{\mathbf{P}^N}(\delta))$ and $\mathbf{C}^{\mathbf{I}_J} \cong H^0(\mathbf{P}_J, \mathscr{O}_{\mathbf{P}_J}(\delta))$. Observe that $\widetilde{\mathscr{Y}}_{1,J}$ and $\widetilde{\mathscr{Y}}_{2,J}$ are both smooth. Indeed, $\widetilde{\mathscr{Y}}_{1,J}$ is isomorphic to the total space of the vector bundle $S|_{\mathbf{P}_J}^{\oplus(k+1)}$ and $\widetilde{\mathscr{Y}}_{2,J}$ is the total space of the vector bundle $S_J^{\oplus(k+1)}$, where $S_J = \ker(H^0(\mathbf{P}_J, \mathscr{O}_{\mathbf{P}_J}(\delta)) \otimes \mathscr{O}_{\mathbf{P}_J} \to \mathscr{O}_{\mathbf{P}_J}(\delta))$. By analogy with $\mathbf{G}_J^{\infty}$, let us denote by $\mathbf{V}_{1,J}^{\infty}$ (resp. $\mathbf{V}_{2,J}^{\infty}$) the set of elements in $(\mathbf{C}^{\mathbf{I}})^{k+1}$ (resp. $(\mathbf{C}^{\mathbf{I}_J})^{k+1}$) at which the fiber in $\widetilde{\mathscr{Y}}_{1,J}$ (resp. $\widetilde{\mathscr{Y}}_{2,J}$) has a positive dimensional component.

First one checks by a straightforward computation that if one denotes by $\rho_J : (\mathbf{C}^{\mathbf{I}})^{k+1} \to (\mathbf{C}^{\mathbf{I}_J})^{k+1}$ the natural map induced by the restriction from $\mathbf{P}^N$ to $\mathbf{P}_J$, one has

$$\mathbf{V}_{1,J}^{\infty} = \rho_J^{-1}\left(\mathbf{V}_{2,J}^{\infty}\right),$$

simply because for any $[T] \in \mathbf{P}_J$ and any $(c_I)_{I \in \mathbf{I}} \in \mathbf{C}^{\mathbf{I}}$, one has $\sum_{I \in \mathbf{I}} c_I T^I = \sum_{I \in \mathbf{I}_J} c_I T^I$.

Moreover, as a consequence of Lemma 2.3 in Benoist's article [1], one obtains, as proved in Corollary 3.1 of [7], that

$$\operatorname{codim}_{(\mathbf{C}^{\mathbf{I}_J})^{k+1}} \mathbf{V}_{2,J}^{\infty} \geqslant \delta + 1. \tag{37}$$

We are now going to bound the dimension of $\hat{\Phi}^{-1}(\mathbf{G}_{\mathrm{J}}^{\infty}) \cap (\mathbf{A}_{\mathrm{def}} \times \hat{\mathrm{X}}_{k,\mathrm{J}}^{\circ})$. Take $\hat{w}_0 \in \hat{\mathrm{X}}_{k,\mathrm{J}}^{\circ}$ and take $\hat{\varphi}_{\hat{w}_0}$ as in (29). From Lemma 3.6 one obtains that

$$\hat{\Phi}^{-1}\big(\mathbf{G}_{\mathrm{J}}^{\infty}\big) \cap \big(\mathbf{A}_{\mathrm{def}} \times \{\hat{w}_0\}\big) \cong \hat{\varphi}_{\hat{w}_0}^{-1}\big(\mathbf{V}_{1,\mathrm{J}}^{\infty}\big) \cap \mathbf{A}_{\mathrm{def}} = (\rho_{\mathrm{J}} \circ \hat{\varphi}_{\hat{w}_0})^{-1}\big(\mathbf{V}_{2,\mathrm{J}}^{\infty}\big) \cap \mathbf{A}_{\mathrm{def}}.$$

But since $x := \pi_{0,k}(\nu_k(\hat{w}_0)) \in \mathrm{X}_{\mathrm{J}}$, we have $\mathbf{I}_x = \mathbf{I}_{\mathrm{J}}$, hence $\rho_x = \rho_{\mathrm{J}}$. Lemma 3.7 thus implies that

$$\operatorname{rank}(\rho_{\mathrm{J}} \circ \hat{\varphi}_{\hat{w}_0}) = (k+1)\#\mathbf{I}_{\mathrm{J}} = \dim\big(\mathbf{C}^{\mathbf{I}_{\mathrm{J}}}\big)^{k+1}.$$

Therefore

$$\begin{aligned}
&\dim\Big(\hat{\Phi}^{-1}\big(\mathbf{G}_{\mathrm{J}}^{\infty}\big) \cap \big(\mathbf{A}_{\mathrm{def}} \times \{\hat{w}_0\}\big)\Big)\\
&\quad\leqslant \dim(\rho_{\mathrm{J}} \circ \hat{\varphi}_{\hat{w}_0})^{-1}\big(\mathbf{V}_{2,\mathrm{J}}^{\infty}\big) \leqslant \dim \mathbf{V}_{2,\mathrm{J}}^{\infty} + \dim\ker(\rho_{\mathrm{J}} \circ \hat{\varphi}_{\hat{w}_0})\\
&\quad\leqslant \dim\big(\mathbf{C}^{\mathbf{I}_{\mathrm{J}}}\big)^{k+1} - \operatorname{codim}_{(\mathbf{C}^{\mathbf{I}_{\mathrm{J}}})^{k+1}} \mathbf{V}_{2,\mathrm{J}}^{\infty} + \dim \mathbf{A} - \operatorname{rank}(\rho_{\mathrm{J}} \circ \hat{\varphi}_{\hat{w}_0})\\
&\quad= \dim \mathbf{A} - \operatorname{codim}_{(\mathbf{C}^{\mathbf{I}_{\mathrm{J}}})^{k+1}} \mathbf{V}_{2,\mathrm{J}}^{\infty}.
\end{aligned}$$

A final computation then yields

$$\begin{aligned}
&\dim\Big(\hat{\Phi}^{-1}\big(\mathbf{G}_{\mathrm{J}}^{\infty}\big) \cap (\mathbf{A}_{\mathrm{def}} \times \hat{\mathrm{X}}_{k,\mathrm{J}})\Big)\\
&\quad\leqslant \dim \mathbf{A} - \operatorname{codim}_{(\mathbf{C}^{\mathbf{I}_{\mathrm{J}}})^{k+1}} \mathbf{V}_{2,\mathrm{J}}^{\infty} + \dim \hat{\mathrm{X}}_{k,\mathrm{J}} < \dim \mathbf{A}
\end{aligned}$$

in view of (37) and our hypothesis on δ. It now suffices to set $\mathbf{A}_{\mathrm{J}} := \mathbf{A}_{\mathrm{def}} \setminus \overline{\mathrm{pr}_1(\hat{\Phi}^{-1}(\mathbf{G}_{\mathrm{J}}^{\infty}) \cap (\mathbf{A}_{\mathrm{def}} \times \hat{\mathrm{X}}_{k,\mathrm{J}}))}$. □

Remark **3.12.** — Observe that this proof shows that the conclusion of Lemma 3.11 would still hold if the condition on δ is replaced by the condition

$$\operatorname{codim}_{(\mathbf{C}^{\mathbf{I}_{\mathrm{J}}})^{k+1}} \mathbf{V}_{2,\mathrm{J}}^{\infty} > \dim \hat{\mathrm{X}}_{k,\mathrm{J}}. \tag{38}$$

3.4. *Proof of Theorem 3.2.* — We are now in position to prove Theorem 3.2. Take $\mathrm{N} \geqslant n$, $k \geqslant \mathrm{N}-1$, $\varepsilon \geqslant m_{\infty}$ and $\delta \geqslant n(k+1) \geqslant n + k(n-1) = \dim \hat{\mathrm{X}}_k$ (or such that δ satisfies (34) and (38) for any J). Let us denote by $\mathscr{Q}$ the (very ample) Plücker line bundle on $\mathrm{Gr}_{k+1}(\mathbf{C}^{\mathbf{I}})$. Let us also denote, for any $\mathrm{J} \subset \{0, \dots, \mathrm{N}\}$, by q_1 and q_2 the canonical projections from $\mathrm{Gr}_{k+1}(\mathbf{C}^{\mathbf{I}}) \times \mathbf{P}_{\mathrm{J}}$ to each factor. The ambiguity of the notation for q_2 should not lead to any confusion. By the definition of $\hat{\Psi}$, one obtains that for any $m \in \mathbf{N}$,

$$\hat{\Psi}^*\big(q_1^*\mathscr{Q}^m \otimes q_2^*\mathscr{O}_{\mathbf{P}^{\mathrm{N}}}(-1)\big) = \nu_k^*\Big(\mathscr{O}_{\mathrm{X}_k}\big(mk'\big) \otimes \pi_{0,k}^*\mathrm{A}^{m(k+1)(u\varepsilon+kv\delta)-vr}\Big) \otimes \mathscr{O}_{\hat{\mathrm{X}}_k}(-m\mathrm{F}). \tag{39}$$

Here we took q_2 for $\mathrm{J} = \varnothing$. The key point in this equality of line bundles is the isolated term $-vr$ which appears on the exponent of the line bundle A.

For any $\mathrm{J} \subset \{0, \dots, \mathrm{N}\}$, by Nakamaye's theorem on the augmented base locus [45], and the definition of $\mathrm{E_J}$, one obtains that $\mathrm{E_J}$ is precisely the augmented base locus $\mathbf{B}_+(q_1^*\mathscr{Q}|_{\mathscr{Y}_\mathrm{J}})$ of $q_1^*\mathscr{Q}|_{\mathscr{Y}_\mathrm{J}}$. Observe that $q_1^*\mathscr{Q}$ and $q_2^*\mathscr{O}_{\mathbf{P}_\mathrm{J}}(1)$ are globally generated and that moreover $q_1^*\mathscr{Q} \otimes q_2^*\mathscr{O}_{\mathbf{P}_\mathrm{J}}(1)$ is ample (it is in fact very ample since, up to the composition with the Plücker embedding, it is the pullback of a tautological very ample line bundle under the Segre embedding for $\mathrm{P}(\Lambda^{k+1}\mathbf{C}^{\mathbf{I}}) \times \mathbf{P}_\mathrm{J}$). Therefore, one obtains from the definition of $\mathbf{B}_+$, by noetherianity, that there exists $m_\mathrm{J} \in \mathbf{N}$ such that

$$\mathrm{E_J} = \mathbf{B}_+\big(q_1^*\mathscr{Q}|_{\mathscr{Y}_\mathrm{J}}\big) = \mathrm{Bs}\big(q_1^*\mathscr{Q}^m \otimes q_2^*\mathscr{O}_{\mathbf{P}_\mathrm{J}}(-1)|_{\mathscr{Y}_\mathrm{J}}\big), \quad \forall m \geqslant m_\mathrm{J}. \tag{40}$$

Set $\mathrm{M} := \max\{m_\mathrm{J} \mid \mathrm{J} \subset \{0, \dots, \mathrm{N}\}\}$. Observe that M only depends on N, k, δ, and define

$$r(v, u, \mathrm{M}, \mathrm{N}, k, \varepsilon, \delta) := \left\lceil \frac{\mathrm{M}(k+1)(u\varepsilon + kv\delta) + 1}{v} \right\rceil \quad \text{and}$$

$$\mathbf{A}_{\mathrm{nef}} := \bigcap_{\mathrm{J} \subset \{0,\dots,\mathrm{N}\}} \mathbf{A}_\mathrm{J} \cap \mathbf{A}^\circ_{\mathrm{def}}.$$

Let us prove that the conclusion of Theorem 3.2 is then satisfied. Take $\mathbf{a} \in \mathbf{A}_{\mathrm{nef}}$. We aim to prove that

$$\nu_k^*\big(\mathscr{O}_{\mathrm{X}_k}(\mathrm{M}k') \otimes \pi_{0,k}^*\mathrm{A}^{-1}\big) \otimes \mathscr{O}_{\hat{\mathrm{X}}_k}(-\mathrm{MF})|_{\hat{\mathrm{H}}_{k,\mathbf{a}}}$$

is nef on $\hat{\mathrm{H}}_{k,\mathbf{a}} \subset \hat{\mathrm{X}}_k$. Take an irreducible curve $\mathrm{C} \subset \hat{\mathrm{H}}_{k,\mathbf{a}}$ and take (the unique) $\mathrm{J} \subset \{0, \dots, \mathrm{N}\}$ such that $\hat{\mathrm{X}}_{k,\mathrm{J}} \cap \mathrm{C} = \mathrm{C}^\circ$ is a non-empty open subset of C. Therefore $\mathrm{C}^\circ \subset \hat{\mathscr{H}}^{\mathrm{rel}}_{k,\mathrm{J}}$, and by Proposition 3.10, $\hat{\Psi}|_{\mathrm{C}^\circ}$ factors through $\mathscr{Y}_\mathrm{J}$. Since $\mathscr{Y}_\mathrm{J}$ is proper, $\hat{\Psi}|_\mathrm{C}$ factors through $\mathscr{Y}_\mathrm{J}$ as well. From Lemma 3.11, one obtains that $\hat{\Phi}(\mathrm{C}^\circ) \cap \mathbf{G}_\mathrm{J}^\infty = \varnothing$ and that therefore $\hat{\Psi}(\mathrm{C}^\circ) \cap \mathrm{E_J} = \varnothing$. In particular,

$$\hat{\Psi}(\mathrm{C}) \not\subset \mathrm{E_J}.$$

From this, and (40), it follows that $\hat{\Psi}(\mathrm{C}) \cdot (q_1^*\mathscr{Q}^\mathrm{M} \otimes q_2^*\mathscr{O}_{\mathbf{P}_\mathrm{J}}(-1)) \geqslant 0$ and that therefore

$$\mathrm{C} \cdot \hat{\Psi}^*\big(q_1^*\mathscr{Q}^\mathrm{M} \otimes q_2^*\mathscr{O}_{\mathbf{P}^\mathrm{N}}(-1)\big) \geqslant 0.$$

Combining this equality with our hypothesis $r \geqslant r(v, u, \mathrm{M}, \mathrm{N}, k, \varepsilon, \delta)$, (39) and the fact that $\nu_k^*\pi_{0,k}^*\mathrm{A}$ is nef, it follows that

$$\mathrm{C} \cdot \big(\nu_k^*\big(\mathscr{O}_{\mathrm{X}_k}(\mathrm{M}k') \otimes \pi_{0,k}^*\mathrm{A}^{-1}\big) \otimes \mathscr{O}_{\hat{\mathrm{X}}_k}(-\mathrm{MF})\big) \geqslant 0.$$

This proves the desired nefness and concludes the proof of Theorem 3.2.

Remark **3.13.** — In order to obtain an effective bound on the degree appearing in the main result, on needs to give an effective bound on M and therefore on the different m_J. Ya Deng was able to provide the effective upper bound δ^k for m_J if instead of requiring (40), one only requires $\mathrm{Bs}(q_1^*\mathscr{Q}^m \otimes q_2^*\mathscr{O}_{\mathbf{P}_J}(-1)|_{\mathscr{Y}_J}) \subseteq q_1^{-1}(\mathbf{G}_J^\infty)$ for all $m \geqslant m_J$. This last condition is weaker than (40), but is sufficient for our purpose. To prove this result, Ya Deng constructs global sections of $q_1^*\mathscr{Q}^m \otimes q_2^*\mathscr{O}_{\mathbf{P}_J}(-1)|_{\mathscr{Y}_J}$ by a geometric argument that allows him to control their base locus.

Let us conclude this section by summarizing the order in which the different constants that appear in the argument have to be chosen and by emphasizing the interdependencies between them. First fix X and the ample line bundle A. Then set $n = \dim X$. Take v_0 such that A^v is very ample for $v \geqslant v_0$; this number only depends on A. Fix $u, v \geqslant v_0$. Fix $\varepsilon \geqslant m_\infty(X_k, A^u)$; this last number depends on k, u and the geometry of X and A (in fact, Ya Deng's result ensures that on can just take $\varepsilon \geqslant k$). Take $\delta \geqslant n(k+1)$. Then, for any $J \subset \{0, \dots, N\}$, take m_J satisfying (40); this number, being determined by the geometry of $\mathscr{Y}_J$, depends only on N, k and δ. Lastly, take $M = M(N, k, \delta) = \max_J\{m_J\}$ and $r \geqslant r(v, u, M, N, k, \varepsilon, \delta)$ defined as above. We emphasize that the most critical point here is that r can to be chosen last, and that in particular M does not depend on r.

Acknowledgements

We warmly thank Lionel Darondeau for his support, his help and the conversations we had. We gratefully thank Olivier Benoist for his help and for the insightful suggestions he provided. We thank Simone Diverio for his useful comments on the present work, as well as Carlo Gasbarri for the discussions we had and for the simplifications he pointed out. We also thank Ya Deng for the improvements he suggested. Lastly, we would like to thank the anonymous referee for helpful suggestions on the manuscript.

REFERENCES

1. O. Benoist, Le théorème de Bertini en famille, *Bull. Soc. Math. Fr.*, **139** (2011), 555–569.
2. G. Berczi, Towards the Green-Griffiths-Lang conjecture via equivariant localisation. ArXiv e-prints, 2015.
3. R. Brody, Compact manifolds and hyperbolicity, *Trans. Am. Math. Soc.*, **235** (1978), 213–219.
4. R. Brody and M. Green, A family of smooth hyperbolic hypersurfaces in P_3, *Duke Math. J.*, **44** (1977), 873–874.
5. D. Brotbek, Hyperbolicity related problems for complete intersection varieties, *Compos. Math.*, **150** (2014), 369–395.
6. D. Brotbek, Symmetric differential forms on complete intersection varieties and applications, *Math. Ann.*, **366** (2016), 417–446.
7. D. Brotbek and L. Darondeau, Complete intersection varieties with ample cotangent bundles. ArXiv e-prints, 2015.
8. H. Clemens, Curves on generic hypersurfaces, *Ann. Sci. Éc. Norm. Super. (4)*, **19** (1986), 629–636.
9. L. Darondeau, On the logarithmic Green–Griffiths conjecture, *Int. Math. Res. Not.*, **2016** (2016), 1871–1923.
10. L. Darondeau, Slanted vector fields for jet spaces, *Math. Z.*, **282** (2016), 547–575.
11. O. Debarre, Varieties with ample cotangent bundle, *Compos. Math.*, **141** (2005), 1445–1459.
12. J.-P. Demailly, Recent progress towards the Kobayashi and Green-Griffiths-Lang conjectures. Manuscript Institut Fourier. Available at: http://www-fourier.ujf-grenoble.fr/~demailly/preprints.html.

13. J.-P. Demailly, Holomorphic Morse inequalities, in *Several Complex Variables and Complex Geometry, Part 2* (*Santa Cruz, CA, 1989*), Proc. Sympos. Pure Math., vol. 52, pp. 93–114, Am. Math. Soc., Providence, 1991.
14. J.-P. Demailly, Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials, in *Algebraic Geometry* (*Santa Cruz, 1995*), Proc. Sympos. Pure Math., vol. 62, pp. 285–360, Am. Math. Soc., Providence, 1997.
15. J.-P. Demailly, Variétés hyperboliques et équations différentielles algébriques, *Gaz. Math.*, **73** (1997), 3–23.
16. J.-P. Demailly, Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture, *Pure Appl. Math. Q.*, **7** (2011), 1165–1207, Special Issue: In memory of Eckart Viehweg.
17. J.-P. Demailly, Towards the Green-Griffiths-Lang conjecture. ArXiv e-prints, 2014.
18. J.-P. Demailly, Proof of the Kobayashi conjecture on the hyperbolicity of very general hypersurfaces. ArXiv e-prints, 2015.
19. J.-P. Demailly and J. El Goul, Connexions méromorphes projectives partielles et variétés algébriques hyperboliques, *C. R. Acad. Sci. Paris, Sér. I, Math.*, **324** (1997), 1385–1390.
20. J.-P. Demailly and J. El Goul, Hyperbolicity of generic surfaces of high degree in projective 3-space, *Am. J. Math.*, **122** (2000), 515–546.
21. Y. Deng, Effectivity in the Hyperbolicity-related problems. ArXiv e-prints, 2016.
22. Y. Deng, On the Diverio-Trapani Conjecture. ArXiv e-prints, 2017.
23. S. Diverio, Differential equations on complex projective hypersurfaces of low dimension, *Compos. Math.*, **144** (2008), 920–932.
24. S. Diverio, Existence of global invariant jet differentials on projective hypersurfaces of high degree, *Math. Ann.*, **344** (2009), 293–315.
25. S. Diverio, J. Merker and E. Rousseau, Effective algebraic degeneracy, *Invent. Math.*, **180** (2010), 161–223.
26. S. Diverio and E. Rousseau, The exceptional set and the Green-Griffiths locus do not always coincide, *Enseign. Math.*, **61** (2015), 417–452.
27. S. Diverio and S. Trapani, A remark on the codimension of the Green-Griffiths locus of generic projective hypersurfaces of high degree, *J. Reine Angew. Math.*, **649** (2010), 55–61.
28. L. Ein, Subvarieties of generic complete intersections, *Invent. Math.*, **94** (1988), 163–169.
29. L. Ein, Subvarieties of generic complete intersections. II, *Math. Ann.*, **289** (1991), 465–471.
30. J. El Goul, Algebraic families of smooth hyperbolic surfaces of low degree in $\mathbf{P}^3_{\mathbf{C}}$, *Manuscr. Math.*, **90** (1996), 521–532.
31. G. Galbura, Il wronskiano di un sistema di sezioni di un fibrato vettoriale di rango i sopra una curva algebrica ed il relativo divisore di Brill-Severi, *Ann. Mat. Pura Appl. (4)*, **98** (1974), 349–355.
32. M. Green and P. Griffiths, Two applications of algebraic geometry to entire holomorphic mappings, in *The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979)*, pp. 41–74, Springer, New York, 1980.
33. J. Harris, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 133, Springer, New York, 1995. A first course, Corrected reprint of the 1992 original.
34. S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings*, Pure and Applied Mathematics, vol. 2, Dekker, New York, 1970.
35. S. Kobayashi, *Hyperbolic Complex Spaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 318, Springer, Berlin, 1998.
36. J. Kollár, *Lectures on Resolution of Singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, 2007.
37. D. Laksov, Wronskians and Plücker formulas for linear systems on curves, *Ann. Sci. Éc. Norm. Supér. (4)*, **17** (1984), 45–66.
38. S. Lang, Hyperbolic and Diophantine analysis, *Bull., New Ser., Am. Math. Soc.*, **14** (1986), 159–205.
39. K. Masuda and J. Noguchi, A construction of hyperbolic hypersurface of $\mathbf{P}^n(\mathbf{C})$, *Math. Ann.*, **304** (1996), 339–362.
40. M. McQuillan, Diophantine approximations and foliations, *Publ. Math. IHÉS*, **87** (1998), 121–174.
41. M. McQuillan, Holomorphic curves on hyperplane sections of 3-folds, *Geom. Funct. Anal.*, **9** (1999), 370–392.
42. J. Merker, Low pole order frames on vertical jets of the universal hypersurface, *Ann. Inst. Fourier (Grenoble)*, **59** (2009), 1077–1104.
43. J. Merker, Algebraic differential equations for entire holomorphic curves in projective hypersurfaces of general type: Optimal lower degree bound, in *Geometry and Analysis on Manifolds*, Progr. Math., vol. 308, pp. 41–142, Birkhäuser/Springer, Cham, 2015.
44. A. M. Nadel, Hyperbolic surfaces in $\mathbf{P}^3$, *Duke Math. J.*, **58** (1989), 749–771.
45. M. Nakamaye, Stable base loci of linear series, *Math. Ann.*, **318** (2000), 837–847.
46. J. Noguchi, Meromorphic mappings into a compact complex space, *Hiroshima Math. J.*, **7** (1977), 411–425.

47. J. Noguchi, Nevanlinna-Cartan theory over function fields and a Diophantine equation, *J. Reine Angew. Math.*, **487** (1997), 61–83.
48. J. Noguchi, Connections and the second main theorem for holomorphic curves, *J. Math. Sci. Univ. Tokyo*, **18** (2011), 155–180.
49. J. Noguchi and J. Winkelmann, *Nevanlinna Theory in Several Complex Variables and Diophantine Approximation*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 350, Springer, Tokyo, 2014.
50. G. Pacienza, Subvarieties of general type on a general projective hypersurface, *Trans. Am. Math. Soc.*, **356** (2004), 2649–2661 (electronic).
51. M. Păun, Vector fields on the total space of hypersurfaces in the projective space and hyperbolicity, *Math. Ann.*, **340** (2008), 875–892.
52. M. Păun, Techniques de construction de différentielles holomorphes et hyperbolicité (d'après J.-P. Demailly, S. Diverio, J. Merker, E. Rousseau, Y.-T. Siu...), *Astérisque*, **361** (2014), 77–113, Exp. No. 1061, vii.
53. E. Rousseau, Équations différentielles sur les hypersurfaces de $\mathbf{P}^4$, *J. Math. Pures Appl. (9)*, **86** (2006), 322–341.
54. E. Rousseau, Weak analytic hyperbolicity of generic hypersurfaces of high degree in $\mathbf{P}^4$, *Ann. Fac. Sci. Toulouse Math. (6)*, **16** (2007), 369–383.
55. J. Greenlees Semple, Some investigations in the geometry of curve and surface elements, *Proc. Lond. Math. Soc.*, **3** (1954), 24–49.
56. B. Shiffman and M. Zaidenberg, Hyperbolic hypersurfaces in $\mathbf{P}^n$ of Fermat-Waring type, *Proc. Am. Math. Soc.*, **130** (2002), 2031–2035 (electronic).
57. Y. Siu, Noether-Lasker decomposition of coherent analytic subsheaves, *Trans. Am. Math. Soc.*, **135** (1969), 375–385.
58. Y. Tong Siu, Defect relations for holomorphic maps between spaces of different dimensions, *Duke Math. J.*, **55** (1987), 213–251.
59. Y.-T. Siu, Hyperbolicity in complex geometry, in *The Legacy of Niels Henrik Abel*, pp. 543–566, Springer, Berlin, 2004.
60. Y.-T. Siu, Hyperbolicity of generic high-degree hypersurfaces in complex projective space, *Invent. Math.*, **202** (2015), 1069–1166.
61. Y.-T. Siu and S.-K. Yeung, Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees, *Am. J. Math.*, **119** (1997), 1139–1172.
62. C. Voisin, On a conjecture of Clemens on rational curves on hypersurfaces, *J. Differ. Geom.*, **44** (1996), 200–213.
63. S.-Y. Xie, On the ampleness of the cotangent bundles of complete intersections, *ArXiv e-prints* 1510.06323, 2015.
64. S.-Y. Xie, Generalized Brotbek's symmetric differential forms and applications. ArXiv e-prints, 2016.
65. Z. Mikhail, The complement to a general hypersurface of degree $2n$ in $\mathbf{CP}^n$ is not hyperbolic, *Sib. Mat. Zh.*, **28** (1987), 91–100, 222.
66. Z. Mikhail, Hyperbolic surfaces in $\mathbf{P}^3$: Examples, *ArXiv Mathematics e-prints*, 2003.

D. B.
Institut de Recherche Mathématique Avancée,
Université de Strasbourg,
Strasbourg, France
brotbek@math.unistra.fr

Manuscrit reçu le 21 juin 2016
Manuscrit accepté le 7 juin 2017
publié en ligne le 27 juin 2017.