FINITE BASIS FOR ANALYTIC MULTIPLE GAPS

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ABSTRACT

An n-gap consists of n many pairwise orthogonal families of subsets of a countable set that cannot be separated. We prove that for every positive integer n there is a finite basis for the class of analytic n-gaps. The proof requires an analysis of certain combinatorial problems on the n-adic tree, and in particular a new partition theorem for trees.

CONTENTS

Introduction

In this paper we are try to provide a structural theory for n-gaps, a notion recently introduced in [2] generalizing the classical notion of gap (here, 2-gap). An n-gap is nothing else than n many families $\Gamma_0, \dots, \Gamma_{n-1}$ of subsets of a fixed countable set N which are hereditary (if $a \in \Gamma_i$ and $b \subset a$, then $b \in \Gamma_i$), pairwise orthogonal (if $a \in \Gamma_i$ and $b \in \Gamma_j$ for $i \neq j$, then $a \cap b$ is finite), and satisfy the non-triviality condition of not being *separated*. Here, separated would mean that we can write $N = a_0 \cup \dots \cup a_{n-1}$ in such a way that Γ_i is trivial on a_i in the sense that $\Gamma_i|_{a_i} = \{a \in \Gamma_i : a \subset a_i\}$ consists only of finite sets. An n-gap may arise in any situation where we have a sequence and different classes of subsequences. For example, N could be a sequence of points in a topological space X, and Γ_i could be the fami ly of su bsequences of N whose cluster points lie in certain set $X_i \subset X$. Another example is $N = \{x_k : k < \omega\}$ a sequence of vectors in a normed space, and each family Γ_i consists of subsequences with a given estimate on the computation of norms of linear combinations (like being c_0-sequence, ℓ_p-sequence, etc.).

The main problem that we address is the following:

Problem **1.** — Given an n-gap $\Gamma = \{\Gamma_0, \dots, \Gamma_{n-1}\}$, can we find an infinite set $M \subset N$ such that the restriction $\{\Gamma_0|_M, \dots, \Gamma_{n-1}|_M\}$ becomes a canonical object?

That this project, even in the case $n = 2$, is of a great complexity was first realized by Hausdorff [14, 15] more than a century ago after a series of earlier papers of Du Bois-Reymond [9] and Hadamard [12] showing that the case of countably generated families

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presents no difficulties. What Hausdorff showed is that there exist 2-gaps that could code objects far outside the reach of structure theory we would hope to develop. It is interesting that even such a non-structure theorem of Hausdorff would bare some fruits such as, for example, the solution of Kaplanski's problem (see [16]) about the automatic continuity in the context of Banach algebras where the answers essentially depended on whether the list of spectra of 2-gaps discovered by Hausdorff [14] is complete or not (see [10], [5], [6]). We were motivated to consider Problem 1 by a more concrete question that arose in [2]:

Problem **2.** — If we are given a 3-gap $\{\Gamma_0, \Gamma_1, \Gamma_2\}$, can we find an infinite set $M \subset N$ and some reordering $\{0, 1, 2\} = \{i, j, k\}$ such that $\{\Gamma_i|_M, \Gamma_j|_M\}$ is still a 2-gap, but $\Gamma_k|_M$ consists only of finite sets?

We know that the answer is, in general, negative, but the counterexample provided in [2] is done in a non-constructive way, using the axiom of choice to produce non-measurable sets. So what if the families Γ_i are, for example, analytic? It was already realized in [19], in the classical context of 2-gaps, that many of the pathologies that might occur in arbitrary gaps can be ruled out for gaps of certain descriptive complexity, that have a more rigid structural theory. In this paper, we develop a structural theory for analytic n-gaps, that not only extends the one found in [19] for 2-gaps, but it provides new information even in the classical setting of two families. Let us mention that the results from [19] have already found some substantial applications (see, for example, [20], [1], [8], [7]).

The main result is that for every natural number $n \geq 2$, there exists a *finite basis* of analytic n-gaps, so that given any analytic n-gap $\{\Gamma_0, \dots, \Gamma_{n-1}\}$ there exists a restriction $\{\Gamma_0|_M, \dots, \Gamma_{n-1}|_M\}$ to an infinite set $M \subset N$ which is an n-gap *isomorphic* (in a sense) to one n-gap from the finite basis. This is the foundation of the theory that allows to give a positive answer to Problem 2 above for analytic n-gaps. The proofs required for this and other concrete facts of the like require considerable extra effort and long computations and this will appear elsewhere.

We achieve our goal in two steps. First, we reduce the problem from n-gaps on an arbitrary set N to n-gaps on the n-adic tree $n^{<\omega}$ (the set of all finite sequences of numbers from $\{0, \dots, n-1\}$). Thus, given any analytic n-gap $\{\Gamma_0, \dots, \Gamma_{n-1}\}$, Theorem 4.2 states that we can find an injective map $u : n^{<\omega} \longrightarrow N$ which, after a permutation, sends $[i]$-chains to sets in Γ_i, for $i = 0, \dots, n-1$. The $[i]$-chains are a special kind of subsets $\{t_0, t_1, \dots\} \subset n^{<\omega}$ in which t_{k+1} is obtained by adding to the right of t_k a sequence of the form $(i, j_1, \dots, j_p)$ with $j_1 \leq i, \dots, j_p \leq i$. The families $\mathcal{C}_i$ of $[i]$-chains constitute an analytic n-gap, and Theorem 4.2 is telling that any other analytic n-gap contains this canonical example inside. The case $n = 2$ of this result is essentially a convenient reformulation of [19, Theorem 3].

In the second step we make a deeper analysis of the combinatorics of the n-adic tree, expanding the classes of $[i]$-chains to a wider collection of *types* of subsets of $n^{<\omega}$

including further kinds of *chains* as well as so-called *combs*. The main tool needed is a new partition principle, Theorem 2.5, that states that if we color all subsets of $n^{<\omega}$ of a given *type* into finitely many colors in a measurable way, then we can pass to a suitable subtree where all sets have the same color. We believe that this partition theorem is of independent interest an may have further applications. The result is given in a similar context as Milliken's theorem [18], but we code more information by demanding to *stay below number i once we get a number i*.

Although all results are stated for analytic n-gaps in order to hold in ZFC, it is still possible to get analogous statements for more general projective n-gaps assuming projective determinacy. This is discussed in Section 5.

We should mention our previous paper [3] as an important precedent of this work. We consider there similar problems but dealing with *countable separation* instead of *separation,* and *strong n-gaps* instead of general n-gaps. The main scheme is similar to the one presented here but far simpler, and in particular no new partition theorem was necessary as Milliken's theorem was enough.

1. Basic definitions

As mentioned above, we introduced the notion of multiple gaps in our previous papers [2, 3]. This time, we shall modify slightly our working definition to gain some generality, we shall work with *preideals* instead of ideals.

Definition **1.1.** — A preideal on a countable set N is a family I of subsets of N such that if $x \in I$ and $y \subset x$ is infinite, then $y \in I$.

The preideal ideal I is analytic if it is analytic as a subset of $\mathcal{P}(N) = 2^N$. We use the symbol $*$ to denote inclusions modulo finite sets, so $x \subset^* y$ means that $x \setminus y$ is finite, and $x =^* y$ means that $x \subset^* y$ and $y \subset^* x$. We say that x and y are orthogonal if $x \cap y =^* \emptyset$. Following the tradition in set-theory, we identify each natural number with its set of predecessors, so that $n = \{0, 1, \dots, n-1\}$. The set of natural numbers is written as $\omega = \{0, 1, 2, \dots\}$. In this way, $\{x_i : i \in n\} = \{x_0, x_1, \dots, x_{n-1}\}$, while $\{x_n\}_{n<\omega}$ denotes an infinite sequence. The letters N and M will denote some general countable infinite sets.

Definition **1.2.** — Let $\Gamma = \{\Gamma_i : i \in n\}$ be a family of n many preideals on the set N and let $\mathfrak{X}$ be a family of subsets of n.

(1) We say that Γ is separated if there exist subsets $a_0, \dots, a_{n-1} \subset N$ such that $\bigcap_{i\in n} a_i = \emptyset$ and $x \subset^* a_i$ for all $x \in \Gamma_i$, $i \in n$.
(2) We say that Γ is an $\mathfrak{X}$-gap if it is not separated, but $\bigcap_{i\in A} x_i =^* \emptyset$ whenever $x_i \in \Gamma_i$, $A \in \mathfrak{X}$.

We say that Γ is analytic if each Γ_i is analytic. In the same way, we can say that Γ is Borel, coanalytic, projective, etc. We will consider only two choices of the family $\mathfrak{X}$, when $\mathfrak{X} = [n]^2$ is the family of all subsets of n of cardinality 2, a $[n]^2$-gap will be called an n-gap, while when $\mathfrak{X}$ consists only of the total set $n = \{0, \ldots, n-1\}$, then an $\mathfrak{X}$-gap will be called an n_*-gap. The notion of n_*-gap is more general than that of a n-gap, since it does not require the preideals to be pairwise orthogonal. On the other hand, the use of n-gaps is more natural in some contexts, and for many of the problems that we discuss here, questions about n_*-gaps can be reduced to questions about n-gaps.

In the language of sequences and subsequences that we used in the introduction, if we have an infinite sequence $\{x_n\}_{n<\omega}$, and $\mathcal{C}$ is a hereditary class of subsequences, then $\mathrm{I} = \{a \subset \omega : \{x_n\}_{n\in a} \in \mathcal{C}\}$ is a preideal. When we talk about analytic, Borel or projective classes, we mean that the corresponding preideals have that complexity. It is a simple exercise that the notion of separation of preideals stated above is equivalent to the notion of separation of classes that was stated in the introduction. In this way, all the results that we shall produce about gaps can be restated as facts about classes of subsequences and the ways that they can be separated.

Remember that the general question that our theory deals with is the following: Given a gap Γ on N, can we find an infinite set $\mathrm{M} \subset \mathrm{N}$ such that the restriction of Γ to M becomes a gap which is canonical in some sense? The restriction of a preideal I to M is the preideal $\mathrm{I}|_{\mathrm{M}} = \{x \in \mathrm{I} : x \subset \mathrm{M}\}$, and the restriction of a gap Γ is $\Gamma|_{\mathrm{M}} = \{\Gamma_i|_{\mathrm{M}} : i \in n\}$. Notice that $\Gamma|_{\mathrm{M}}$ may not be in general a gap, as the preideals may become separated when restricted to M.

The orthogonal of I is the family $\mathrm{I}^{\perp}$ consisting of all $x \subset \mathrm{N}$ such that $x \cap y =^* \emptyset$ for all $y \in \mathrm{I}$. The orthogonal of the gap Γ is $\Gamma^{\perp} = (\bigcup_{i\in n} \Gamma_i)^{\perp}$. The gap Γ is called dense if $\Gamma^{\perp}$ is just the family of finite subsets of N.

Definition **1.3.** — Given Γ and Δ two n_*-gaps on countable sets N and M, we say that $\Gamma \leq \Delta$ if there exists a one-to-one map $\phi : \mathrm{N} \longrightarrow \mathrm{M}$ such that for every $i \in n$,

(1) if $x \in \Gamma_i$ then $\phi(x) \in \Delta_i$.
(2) If $x \in \Gamma_i^{\perp}$ then $\phi(x) \in \Delta_i^{\perp}$.

When Γ is a n-gap, the second condition can be substituted by saying that if $x \in \Gamma^{\perp}$ then $\phi(x) \in \Delta^{\perp}$. Notice also that if Δ is a n-gap, Γ is a n_*-gap, and $\Gamma \leq \Delta$, then Γ is an n-gap. Another observation is that the above definition implies that $\phi(x) \in \Delta_i^{\perp\perp}$ if and only if $x \in \Gamma_i^{\perp\perp}$, and $\phi(x) \in \Delta^{\perp}$ if and only if $x \in \Gamma^{\perp}$. Therefore the gaps $\{\Gamma_i^{\perp\perp} : i < n\}$ and $\{\Delta_i^{\perp\perp}|_{\phi(\mathrm{N})} : i < n\}$ are completely identified under the bijection $\phi : \mathrm{N} \longrightarrow \phi(\mathrm{N})$.

Definition **1.4.** — An analytic n_*-gap Γ is said to be a minimal analytic n_*-gap if for every other analytic n_*-gap Δ, if $\Delta \leq \Gamma$, then $\Gamma \leq \Delta$.

Definition **1.5.** — Two minimal analytic n_*-gaps Γ and Γ' are called equivalent if $\Gamma \leq \Gamma'$ (hence also $\Gamma' \leq \Gamma$).

In this language, one of our main results can be stated as follows:

Theorem **1.6.** — *Fix a natural number n. For every analytic n_*-gap Γ there exists a minimal analytic n_*-gap Δ such that $\Delta \leq \Gamma$. Moreover, up to equivalence, there exist only finitely many minimal analytic n_*-gaps.*

The same statements hold for n-gaps instead of n_*-gaps, the minimal analytic n-gaps are a subset of the minimal analytic n_*-gaps.

2. A partition theorem

In this section we state and prove a new pigeon hole principle that at the same time incorporates some features of the infinite Hales-Jewett theorem for left-variable words ([13], [4]) and some features of the Gowers theorem for FIN_k ([11]; see also [21]). In particular, we shall rely on the Galvin-Glazer method of idempotent ultrafilters on partial semigroups. We refer the reader to the introductory chapters of [21] where this method is explained in details and where both the Gowers theorem and the extension of the Hales-Jewett theorem are proved using this method. The reader will find there also some details and references about the long and intricate way this subject was developed so that we can comment here only about the new ideas in the proof below. First of all, we have to restrict ourselves here to idempotent ultrafilters $\mathcal{U}_k$ on semigroups of words W_k that besides the usual equations $\mathrm{T}(\mathcal{U}_k) = \mathcal{U}_l$ satisfy the equations $\mathcal{U}_k^\frown \mathcal{U}_l = \mathcal{U}_l$ rather than the stronger equations $\mathcal{U}_k^\frown \mathcal{U}_l = \mathcal{U}_l^\frown \mathcal{U}_k = \mathcal{U}_l$ for $l \leq k$. The idempotent ultrafilters $\mathcal{U}_k$ on W_k are then used to obtain an infinite-dimensional Ramsey statement that involves the notion of an $\mathcal{U}_k$-tree. An infinite sequence $\{w_0, w_1, \ldots\}$ generates a partial subsemigroup $\mathit{Full}(w_0, w_1, \ldots)$ in the standard way (see [21, Section 2.5]). The crucial lemma here is that for every $\mathcal{U}_k$-subtree Υ of $\mathrm{W}_k^{<\omega}$ there exists a rapidly increasing sequence $w_0, w_1, \ldots$ of elements of W_k such that $(x_0, x_1, \ldots, x_n) \in \Upsilon$ for every $x_0, x_1, \ldots, x_n \in \mathit{Full}(w_0, w_1, \ldots)$ with $\lambda(x_0) < \lambda(x_1) < \cdots < \lambda(x_n)$, where for $x \in \mathit{Full}(w_0, w_1, \ldots)$, by $\lambda(x)$, we denote the maximal index of a term of the sequence $\{w_0, w_1, \ldots\}$ that occurs in the unique concatenation that forms x. This allows us to transfer Souslin-measurable colorings of subtrees of W_k of the same shape to colorings of branches of the tree $\mathrm{W}_k^{<\omega}$ and in return get a copy of W_k with all subtrees of the given shape monochromatic. This is quite different from the standard method that involves the Ramsey space of strong subtrees of a given rooted finitely branching tree U of height ω (see [18], [21, Chapter 6]). We expect that this approach will find some other uses.

Given a set A, we denote by $\mathrm{A}^{<\omega}$ the set of all finite sequences of elements of A. Remember that we identify a natural number m with its set of predecessors, $m = \{0, 1, \ldots, m-1\}$. Thus, $m^{<\omega}$ is the m-adic tree. We consider two order relations on $m^{<\omega}$. Consider $t = (t_0, \ldots, t_p)$ and $s = (s_0, \ldots, s_q)$ in $m^{<\omega}$, the tree order is defined by $t < s$ if and only if $p < q$ and $t_i = s_i$ for all $i \leq p$. The linear order relation $\prec$ is given by:

$t \prec s$ if and only if either $(p < q)$ or $(p = q$ and $t_{\min\{i:t_i \neq s_i\}} < s_{\min\{i:t_i \neq s_i\}})$. The concatenation of t and s is $t^\frown s = (t_0, \dots, t_p, s_0, \dots, s_q)$. We denote by $t \wedge s$ the infimum of t and s in the order $<$, that is, $t \wedge s$ is the largest common initial segment of t and s. If $t^\frown s = r$, then we write $s = r \setminus t$.

For a fixed natural number k, we denote by W_k the set of all finite sequences of natural numbers from $\{0, \dots, k\}$ that start by k, that is

$$W_k = \left\{(t_0, t_1, \dots, t_p) \ : \ t_0 = k, \ t_i \in \{0, \dots, k\}, \ i = 1, \dots, p\right\}$$

We shall view the set W_k as a semigroup, endowed with concatenation $^\frown$ as the operation. Define $T : W_k \longrightarrow W_{k-1}$ by

$$T(w)(i) = \max\left\{0, w(i) - 1\right\}$$

That is, $T(w)$ is a word with the same number of letters as w, and at each place $T(w)$ has a number one unit less than in w, except for zeros which are preserved. Let $T^{(0)} : W_k \longrightarrow W_k$ be the identity map, $T^{(1)} = T$ and $T^{(j)} : W_k \longrightarrow W_{k-j}$ be the j-th iterate of T. We will often denote this iteration as $T_i = T^{(k-i)} : W_k \longrightarrow W_i$ using the same subindex for T_i as for the range space W_i.

Definition **2.1.** — We will say that a subset $F \subset m^{<\omega}$ is *closed* if it satisfies:

(1) If $s, t \in F$, then $s \wedge t \in F$
(2) If $s = t^\frown r_1^\frown \cdots ^\frown r_k$ with $t, s \in F$, $r_1 \in W_{i_1}, \dots, r_k \in W_{i_k}$, $i_1 < i_2 < \cdots < i_k$, then $t^\frown r_1 \in F$ (therefore also $t^\frown r_1^\frown r_2 \in F$, etc.)

Given $F \subset m^{<\omega}$ we will denote by $\langle F \rangle$ the intersection of all closed sets which contain F, which is itself a closed set.

Definition **2.2.** — Consider sets $X \subset m^{<\omega}$, $Y \subset n^{<\omega}$. A function $f : X \longrightarrow Y$ is called an *equivalence* if it is the restriction of a bijection $g : \langle X \rangle \longrightarrow \langle Y \rangle$ satisfying the following

(1) $g(t \wedge s) = g(t) \wedge g(s)$ for all $t, s \in \langle X \rangle$,
(2) $g(t) \prec g(s)$ if and only if $t \prec s$ for all $t, s \in \langle X \rangle$,
(3) For all $t, s \in \langle X \rangle$ with $t \leq s$ and every k, we have that $s \setminus t \in W_k$ if and only if $g(s) \setminus g(t) \in W_k$.

Notice that if s is the immediate successor of t in $\langle X \rangle$ (that is, $t < s$ but there is no $r \in \langle X \rangle$ with $t < r < s$), then $s \setminus t \in W_k$ for some k, and condition (3) of the above definition can be considered just for pairs of immediate successors. The sets X and Y are called equivalent if there is an equivalence between them.

A sequence $\{w_0, w_1, \dots\} \subset \mathrm{W}_k$ is called rapidly increasing if

$$|w_i| > \sum_{j<i} |w_j|$$

for every i. A family $\{w_s : s \in m^{<\omega}\} \subset \mathrm{W}_k$ will be called rapidly increasing if for every $s \in m^{<\omega}$ we have

$$|w_s| > \sum_{t \prec s} |w_t|.$$

Definition **2.3.** — Let $m \leq n < \omega$. A function $\psi : m^{<\omega} \longrightarrow n^{<\omega}$ will be called a nice embedding if there exists a rapidly increasing family $\{w_s : s \in m^{<\omega}\} \subset \mathrm{W}_{m-1}$ such that for every $t \in m^{<\omega}$ and for every $i \in m$, we have that $\psi(t^\frown i) = \psi(t)^\frown \mathrm{T}_i(w_{t^\frown i})$.

Notice that the above implies that ψ is one-to-one. Along this section we are mostly interested in nice embeddings from the m-adic tree into itself. The important thing about nice embeddings is that they preserve equivalence.

Proposition **2.4.** — *If $\psi : m^{<\omega} \longrightarrow n^{<\omega}$ is a nice embedding, then X is equivalent to $\psi(\mathrm{X})$ for every set $\mathrm{X} \subset m^{<\omega}$.*

The range of a nice embedding ψ will be call a nice subtree of $m^{<\omega}$, which is naturally bijected with $m^{<\omega}$ itself by ψ. For a fixed set $\mathrm{X}_0 \subset m^{<\omega}$, let us say that Y is an X_0-set if Y is equivalent to X_0. It is easy to check that the family of all X_0-subsets of $m^{<\omega}$ is closed in the product topology of the Cantor set $2^{m^{<\omega}}$, hence this family has a natural Polish topology. This section is devoted to the proof of the following theorem:

Theorem **2.5.** — *Fix a set $\mathrm{X}_0 \subset m^{<\omega}$. Then for every finite partition of the X_0-subsets of $m^{<\omega}$ into finitely many Suslin-measurable sets, there exists a nice subtree $\mathrm{T} \subset m^{<\omega}$ all of whose X_0-subsets lie in the same piece of the partition.*

By Suslin measurability, we mean with respect to the σ-algebra generated by analytic sets. This is a partition theorem for trees in a similar spirit as Milliken's Theorem [18]. Partition theorems are often stated in the language of *colorings*. Having a finite partition $\mathrm{X} = \bigcup_{i<n} \mathrm{X}_i$ of a set X is equivalent to having a function $c : \mathrm{X} \longrightarrow n$, that is called a coloring, and $c(y)$ is called the color of y. A subset $\mathrm{Y} \subset \mathrm{X}$ lies in one piece of the partition if and only if it is monochromatic for the coloring c, meaning that $\mathrm{Y} \subset c^{-1}(i)$ for some i. The simplest case of Theorem 2.5 happens when X_0 is a singleton:

Corollary **2.6.** — *If we color $n^{<\omega}$ into finitely many colors, then there is a nice subtree which is monochromatic.*

Let W_l^* be the collection of all nonprincipal ultrafilters on W_l. We extend the concatenation $^\frown$ to an operation on W_l^* that we also denote by $^\frown$:

$$A \in \mathcal{U}^\frown\mathcal{V} \Leftrightarrow \left(\mathcal{U}x\ \mathcal{V}y\ x^\frown y \in A\right)$$

This[1] makes $(W_l^*, ^\frown)$ a compact left topological semigroup.[2] For every $\mathcal{U} \in W_k^*$ define

$$T(\mathcal{U}) = \left\{T(X) : X \in \mathcal{U}\right\} = \left\{Y \subset W_{k-1} : T^{-1}(Y) \in \mathcal{U}\right\}$$

Notice that $A \in T(\mathcal{U})$ if and only if $\mathcal{U}x\ T(x) \in A$. The function $T : W_k^* \longrightarrow W_{k-1}^*$ is a continuous onto homomorphism.

We shall construct by induction ultrafilters $\mathcal{U}_k \in W_k^*$ for $k = 0, 1, \ldots$ which will have the following properties:

(1) Each $\mathcal{U}_k$ is a minimal idempotent of W_k^*. Idempotent means that $\mathcal{U}_k{}^\frown\mathcal{U}_k = \mathcal{U}_k$, and $\mathcal{U}_k$ is minimal among the set of idempotents of W_k^* in the order given by $\mathcal{U} \leq \mathcal{V}$ iff $\mathcal{U} = \mathcal{U}^\frown\mathcal{V} = \mathcal{V}^\frown\mathcal{U}$. Cf. [21, Chapter 2].
(2) $T(\mathcal{U}_{k+1}) = \mathcal{U}_k$ for every $k = 0, 1, 2, \ldots$.
(3) $\mathcal{U}_k{}^\frown\mathcal{U}_l = \mathcal{U}_k$ whenever $l \leq k$.

Notice that condition (3) above is just equivalent to $\mathcal{U}_{k+1}{}^\frown\mathcal{U}_k = \mathcal{U}_{k+1}$ for every k. We choose $\mathcal{U}_0$ to be any minimal idempotent of W_0^* (see [21, Lemma 2.2]).

Construction of $\mathcal{U}_{k+1}$ from $\mathcal{U}_k$: Let

$$S = \left\{\mathcal{X} \in W_{k+1}^* : T(\mathcal{X}) = \mathcal{U}_k\right\}.$$

Then S is a closed subsemigroup of W_{k+1}^* and

$$S^\frown\mathcal{U}_k = \left\{\mathcal{X}^\frown\mathcal{U}_k : \mathcal{X} \in S\right\}$$

is a closed left-ideal of S. Using [21, Lemma 2.2] again, we find $\mathcal{U}_{k+1} \in S^\frown\mathcal{U}_k$ a minimal idempotent of $S^\frown\mathcal{U}_k$, which is in turn a minimal idempotent of S. Notice that $\mathcal{U}_{k+1}{}^\frown\mathcal{U}_k = \mathcal{U}_{k+1}$ since $\mathcal{U}_{k+1} \in S^\frown\mathcal{U}_k$ and $\mathcal{U}_k$ is idempotent. Also $T(\mathcal{U}_{k+1}) = \mathcal{U}_k$. It remains to show that $\mathcal{U}_{k+1}$ is a minimal idempotent of W_{k+1}^*. Let $\mathcal{V} \leq \mathcal{U}_{k+1}$ be an idempotent of W_{k+1}^*. Since T is a homomorphism and $T(\mathcal{U}_{k+1}) = \mathcal{U}_k$, we have that $T(\mathcal{V})$ is an idempotent of W_k^* such that $T(\mathcal{V}) \leq \mathcal{U}_k$. Since $\mathcal{U}_k$ was minimal, we conclude that $T(\mathcal{V}) = \mathcal{U}_k$, hence $\mathcal{V} \in S$. Since $\mathcal{U}_{k+1}$ was a minimal idempotent of S we conclude that $\mathcal{U}_{k+1} = \mathcal{V}$.

The construction of the ultrafilters $\mathcal{U}_k$ is thus finished. We define a $\mathcal{U}_k$-tree to be a nonempty downwards closed subtree Υ of $W_k^{<\omega}$ such that

$$\left\{x \in W_k : (x_0, \ldots, x_n, x) \in \Upsilon\right\} \in \mathcal{U}_k$$

[1] The notation $\mathcal{U}x\ P(x)$ means that $\{x : P(x)\} \in \mathcal{U}$.

[2] This means that the operation $\mathcal{U} \mapsto \mathcal{U}^\frown\mathcal{V}$ is continuous for every $\mathcal{V} \in W_l^*$ that we endow with its Stone topology as a set of ultrafilters.

for every $(x_0, \dots, x_n) \in \Upsilon$.

We shall use the following lemma which is a corollary of [21, Theorem 7.42]:

Lemma **2.7.** — *For every finite Suslin-measurable coloring of the branches of* $W_k^{<\omega}$ *there exists a* $\mathcal{U}_k$*-tree* Υ *such that the set of branches of* Υ *is monochromatic.*

Definition **2.8.** — Let $\{w_0, w_1, \dots\}$ be a rapidly increasing sequence of elements of W_k.

$$\mathit{Full}(w_0, w_1, \dots) = \Big\{ w_{m_0}{}^\frown T_{k_1}(w_{m_1})^\frown \cdots {}^\frown T_{k_n}(w_{m_n}) : \\ n < \omega,\ 0 \leq k_1, \dots, k_n \leq k,\ m_0 < \cdots < m_n < \omega \Big\}$$

Given $x = w_{m_0}{}^\frown T_{k_1}(w_{m_1})^\frown \cdots {}^\frown T_{k_n}(w_{m_n})$ as above, we denote $\lambda(x) = m_n$ the last subindex of w_i which appears in the expression of x. Notice that this is properly defined because the sequence $\{w_0, w_1, \dots\}$ is rapidly increasing.

Lemma **2.9.** — *Given* Υ *a* $\mathcal{U}_k$*-tree of* $W_k^{<\omega}$ *there exists a rapidly increasing sequence* $w_0, w_1, \dots$ *of elements of* W_k *such that*

$$(x_0, x_1, \dots, x_n) \in \Upsilon$$

for every $x_0, x_1, \dots, x_n \in \mathit{Full}(w_0, w_1, \dots)$ *with* $\lambda(x_0) < \lambda(x_1) < \cdots < \lambda(x_n)$.

Proof. — For every $\bar{x} = (x_0, \dots, x_n) \in \Upsilon$, let $P_{\bar{x}} = \{x \in W_k : (x_0, \dots, x_n, x)\} \in \Upsilon$. Along this proof we denote $\mathcal{U}_k = \mathcal{U}$. We know that $P_{\bar{x}} \in \mathcal{U}$ for every $\bar{x} \in \Upsilon$. We can assume without loss of generality that $P_{\bar{x}} \supset P_{\bar{y}}$ whenever $x < y$ (in the tree order, meaning that y is an end-extension of x). We shall construct the sequence $w_0, w_1, \dots$ by induction.

Construction of w_0. We know that $\mathcal{U}x \ \ x \in P_\emptyset$, and since

$$\mathcal{U} = \mathcal{U}^\frown T_{k_1}(\mathcal{U})^\frown \cdots {}^\frown T_{k_n}(\mathcal{U})$$

for every $k_1, \dots, k_n \leq k$ we have that

$$\mathcal{U}y_0 \ \ \mathcal{U}y_1 \ \cdots \ \mathcal{U}y_n \quad y_0{}^\frown T_{k_1}(y_1)^\frown \cdots {}^\frown T_{k_n}(y_n) \in P_\emptyset$$

In particular, we can choose $w_0 \in P_\emptyset$ such that

$$(*) \qquad \mathcal{U}y_1 \ \cdots \ \mathcal{U}y_n \quad w_0{}^\frown T_{k_1}(y_1)^\frown \cdots {}^\frown T_{k_n}(y_n) \in P_\emptyset$$

whenever $0 \leq k_0 < k_1 < \cdots < k_n \leq k$, since there are only finitely many choices for indices k_i like this. Notice however, that once w_0 is chosen in this way, the statement $(*)$ above holds whenever $0 \leq k_0, k_1, \dots, k_n \leq k$ (now infinitely many possibilities). The reason is that if we have an expression

$$x = w_0{}^\frown T_{k_1}(y_1)^\frown \cdots {}^\frown T_{k_n}(y_n)$$

we can choose $1 = i_1 < \cdots < i_m$ such that

$$k_1 = k_{i_1} < k_{i_2} < \cdots < k_{i_m} = k_m$$

such that $k_j \leq k_{i_l}$ whenever $i_l \leq j < i_{l+1}$. And then, we can rewrite

$$x = w_0^\frown \mathrm{T}_{k_1}\left(y_1^\frown \mathrm{T}^{(k_1-k_2)}(y_2)^\frown \cdots\right)^\frown \mathrm{T}_{k_{i_2}}\left(y_{k_{i_2}}^\frown \mathrm{T}^{(k_{i_2}-k_{i_2+1})}(y_{k_{i_2+1}})^\frown \cdots\right)^\frown \cdots$$

Construction of w_m. Let $\mathrm{F} = \mathit{Full}(w_0, \ldots, w_{m-1})$ and

$$\mathrm{G} = \left\{\bar{x} = (x_0, \ldots, x_\xi) : x_0, \ldots, x_\xi \in \mathrm{F}, \lambda(x_0) < \cdots < \lambda(x_\xi)\right\}.$$

Notice that G is finite. Our inductive hypotheses are that for every $(x_0, \ldots, x_\xi) \in \mathrm{G}$ and every $0 \leq k_1, \ldots, k_n \leq k$ we have that

$$(*) \qquad x_\xi \in \mathrm{P}_{(x_0,\ldots,x_{\xi-1})}$$

$$(**) \qquad \mathcal{U}y_1 \cdots \mathcal{U}y_n \quad x_\xi^\frown \mathrm{T}_{k_1}(y_1)^\frown \cdots ^\frown \mathrm{T}_{k_n}(y_n) \in \mathrm{P}_{(x_0,\ldots,x_{\xi-1})}$$

The hypothesis $(*)$ will prove the statement of the lemma. The hypothesis $(**)$ is a technical condition necessary for the inductive procedure. On the other hand, like in the case of the construction of w_0, we have that $\mathcal{U}y\ y \in \mathrm{P}_{\bar{x}}$ for every $\bar{x} \in \mathrm{G}$, and this implies that for every $\bar{x} \in \mathrm{G}$ and every $0 \leq k_1, \ldots, k_n \leq k$

$$\mathcal{U}y_0\, \mathcal{U}y_1 \cdots \mathcal{U}y_n \quad y_0^\frown \mathrm{T}_{k_1}(y_1)^\frown \cdots ^\frown \mathrm{T}_{k_n}(y_n) \in \mathrm{P}_{\bar{x}}$$

Therefore in particular, we can find w_m such that for every $k_0 \leq k$, every $0 \leq k_1 < \cdots < k_n \leq k$ and every $\bar{x} = (x_0, \ldots, x_\xi) \in \mathrm{G}$ we have that

$$\begin{aligned}
&w_m \in \mathrm{P}_{\bar{x}}\\
&\mathcal{U}y_1 \cdots \mathcal{U}y_n \quad w_m^\frown \mathrm{T}_{k_1}(y_1)^\frown \cdots ^\frown \mathrm{T}_{k_n}(y_n) \in \mathrm{P}_{\bar{x}}\\
&x_\xi^\frown \mathrm{T}_{k_0}(w_m) \in \mathrm{P}_{(x_0,\ldots,x_{\xi-1})}\\
&\mathcal{U}y_1 \cdots \mathcal{U}y_n \quad x_\xi^\frown \mathrm{T}_{k_0}(w_m)^\frown \mathrm{T}_{k_1}(y_1)^\frown \cdots ^\frown \mathrm{T}_{k_n}(y_n) \in \mathrm{P}_{(x_0,\ldots,x_{\xi-1})}
\end{aligned}$$

By the same trick that we used in the case of the construction of w_0, the above sentences actually hold whenever $0 \leq k_0, k_1, \ldots, k_n \leq k$. This completes the proof, since the statements above imply that the inductive hypotheses $(*)$ and $(**)$ are transferred to the next step. $\square$

We proceed now to the proof of Theorem 2.5. Without loss of generality we can suppose that X_0 is a closed set. Indeed, if X_0 is not closed, consider its closure $\mathrm{Y}_0 = \langle \mathrm{X}_0 \rangle$. If $\mathrm{Y} \sim \mathrm{Y}_0$, then there is a unique set $\mathrm{X} \sim \mathrm{X}_0$ such that $\mathrm{Y} = \langle \mathrm{X} \rangle$, and the correspondence $\mathrm{Y} \leftrightarrow \mathrm{X}$ is Suslin-measurable. In this way, we reduce the general case to the case of closed X_0. We can suppose that X_0 is infinite as well. If we prove the theorem for infinite X_0, the finite case follows as a corollary, just making X_0 infinite by adding zeros above a maximal node. Enumerate $X_0 = \{x_0 \prec x_1 \prec \cdots\}$. Let $k = m-1$, and we consider the infinite product W_k^ω that we identify when convenient with the branches of the tree $W_k^{<\omega}$. Let $\tilde{W}_k^\omega \subset W_k^\omega$ be the set of all sequences which are rapidly increasing. We are going to define a function Φ that associates to each $z \in \tilde{W}_k^\omega$ an X_0-set $\Phi(z) \subset m^{<\omega}$. The set $\Phi(z)$ will be the range of a function $\phi_z : X_0 \longrightarrow m^{<\omega}$ that establishes an equivalence between X_0 and $\Phi(z) = \phi_z(X_0)$. The function ϕ_z is defined inductively. As a starting point of the induction, $\phi_z(x_0) = z_0$. Now, suppose that $\phi_z(x_q)$ is defined for $q < p$ and we shall define $\phi_z(x_p)$. Let x_q be the $\leq$-immediate predecessor of x_p in X_0, and suppose that $x_p = x_q{}^\frown r$ with $r \in W_i$, $i \leq k$. Then, define $\phi_z(x_p) = \phi_z(x_q){}^\frown T_i(z_p)$.

We consider now a finite partition of W_k^ω, in which one piece is the set $W_k^\omega \setminus \tilde{W}_k^\omega$, while the partition of $\tilde{W}_k^\omega$ is induced by the given partition of the X_0-sets of $m^{<\omega}$ through the function Φ. By Lemma 2.7 there exists a $\mathcal{U}_k$-tree $\Upsilon \subset W_k^{<\omega}$ all of whose branches lie in the same piece of the partition. This piece cannot be $W_k^\omega \setminus \tilde{W}_k^\omega$ since every $\mathcal{U}_k$ tree has rapidly increasing branches. So what we have is that for each rapidly increasing branch z of Υ, the set $\Phi(z)$ has the same color.

Let $\{w_0, w_1, \ldots\}$ be the sequence given by Lemma 2.9 applied to the Υ that we found. Let $F = Full(w_0, w_1, \ldots)$. We reorder $\{w_0, w_1, \ldots\}$ in the form of a rapidly increasing family $\{w_s : s \in m^{<\omega}\}$. We claim that the nice embedding that we are looking for is the one given by $\psi(\emptyset) = w_\emptyset$ and $\psi(t^\frown i) = \psi(t)^\frown T_i(w_{t^\frown i})$. In order to check this, it is enough to prove that for every X_0-set Y, the set $\psi(Y)$ is of the form $\Phi(z)$ for some $z \in W_k^\omega$ which is a branch of Υ. So let $Y = \{y_0 \prec y_1 \prec \cdots\}$ be an X_0-set. Let y_i^- be the $\leq$-immediate predecessor of y_i in Y and write $\psi(y_i \setminus y_i^-) = T_j(z_i)$ for some $z_i \in W_k$. Then $z_i \in Full(w_0, w_1, \ldots)$ and $\lambda(z_0) < \lambda(z_1) < \cdots$, so by Lemma 2.9 we have that $(z_0, z_1, \ldots)$ is a branch of Υ. Moreover $\psi(Y) = \Phi(z)$ and this finishes the proof of Theorem 2.5.

We finish this subsection with the following variation of Theorem 2.5. We refer to [17] for information on projective sets and the axiom of projective determinacy.

Theorem **2.10** *(Projective Determinacy). — Fix a set $X_0 \subset m^{<\omega}$. Then for every finite partition of the X_0-subsets of $m^{<\omega}$ into finitely many projective sets, there exists a nice subtree $T \subset m^{<\omega}$ all of whose X_0-subsets lie in the same piece of the partition.*

We do not include the proof here since it is out of the scope of this paper. It follows the general lines of Woodin's proof in [22] that every projective set is Ramsey in the classical sense (cf. [21]) under the assumption of the projective determinacy.

3. Types in the m-adic tree

Among the equivalence classes of subsets of $m^{<\omega}$ to which Theorem 2.5 can be applied, we are a particularly interested in the *minimal* equivalence classes of infinite sets, which are described by what we call *types*.

Definition **3.1.** — A type τ is a triple $\tau = (\tau^0, \tau^1, \lhd)$, where τ_0 and τ_1 are finite subsets of ω with $\tau^0 \neq \emptyset$, $\min(\tau^0) \neq \min(\tau^1)$, together with a linear order relation $\lhd$ on the set $(\tau^0 \times \{0\}) \cup (\tau^1 \times \{1\})$ which extends the natural order of τ^0 and of τ^1 and whose maximum is $(\max(\tau^0), 0)$.

Notice that in the above definition we demand that $\tau^0 \neq \emptyset$, but τ^1 may be empty or not. The sentence "*extends the natural order of τ^0 and of τ^1*" means that $(k, i) \lhd (k', i)$ whenever $k < k'$ and $i \in \{0, 1\}$. A type τ will be represented as a 'matrix' where the lower row is τ^0, the upper row is τ^1 and the order $\lhd$ is read from left to right (so the rightmost element must be always below). For example

$$\tau = \begin{bmatrix} & 6 & & & 9 & \\ 1 & & 3 & 6 & & 7 \end{bmatrix}$$

would represent the type $(\{1, 3, 6, 7\}, \{6, 9\}, \lhd)$ with the order

$$(1, 0) \lhd (6, 1) \lhd (3, 0) \lhd (6, 0) \lhd (9, 1) \lhd (7, 0)$$

When $\tau^1 = \emptyset$ we will write a 'matrix' with just one row.

Definition **3.2.** — Consider a type τ where $\tau^0 = \{k_0 < \cdots < k_n\}$ and $\tau^1 = \{l_0 < \cdots < l_m\}$. We say that a couple (u, v) is a rung of type τ if the following conditions hold:

(1) u can be written as $u_0^\frown \cdots ^\frown u_n$ where $u_i \in W_{k_i}$,
(2) v can be written as $v_0^\frown \cdots ^\frown v_m$ where $v_i \in W_{l_i}$,
(3) $(k_i, 0) \lhd (l_j, 1)$ if and only if $u_0^\frown \cdots ^\frown u_i \prec v_0^\frown \cdots ^\frown v_j$.

In the above definition notice that $v = \emptyset$ if and only if $\tau^1 = \emptyset$.

Definition **3.3.** — Consider a type τ. We say that an infinite set $X \subset m^{<\omega}$ is of type τ if there exists $u \in m^{<\omega}$ and a sequence of rungs $(u_0, v_0), (u_1, v_1), \ldots$ of type τ such that we can write $X = \{x_0, x_1, \ldots\}$ and

$$x_k = u^\frown u_0^\frown u_1^\frown \cdots ^\frown u_{k-1}{}^\frown v_k$$

for $k = 0, 1, \ldots$.

When $\tau^1 = \emptyset$, subsets of type τ will be called τ-chains. If $\tau^1 \neq \emptyset$ they will be called τ-combs. A type in $m^{<\omega}$ is a type such that $\tau^0, \tau^1 \in m^{<\omega}$. These are the possible types of subsets of $m^{<\omega}$.

Let us give a couple of examples as illustration. The set

$$\{(00), (00213), (00213213), (00213213213), \dots\}$$

is a [23]-chain, because if satisfies Definition 3.3 for $u = (00)$, and the rungs (u_i, v_i) where $v_i = \emptyset$, and $u_i = (21)^\frown(3)$, with $(21) \in W_2$ and $(3) \in W_3$.

On the other hand, the set

$$\{(005), (002135), (002132135), (002132132135), \dots\}$$

is a $[{}^5{}_{23}]$-comb, because if satisfies Definition 3.3 for $u = (00)$, and the rungs (u_i, v_i) where $v_i = (5) \in W_5$, and $u_i = (21)^\frown(3)$, with $(21) \in W_2$ and $(3) \in W_3$.

For a fixed type τ, the sets of type τ constitute an equivalence class of subsets of $m^{<\omega}$. Every infinite subset of a set of type τ has again type τ. These facts, together with the following lemma, imply that types can be identified with the *minimal* equivalence classes of infinite subsets of $m^{<\omega}$.

Lemma **3.4.** — *If x is an infinite subset of $m^{<\omega}$, then there exists a type τ and an infinite subset $y \subset x$ of type τ.*

Proof. — Define inductively $\{t_k : k < \omega\} \subset x$, a chain $\{s_k : k < \omega\} \subset m^{<\omega}$ and infinite sets $x = x_0 \supset x_1 \supset \cdots$ in the following way: First, $x_0 = x$, $s_0 = \emptyset$ and $t_0 \in x$ is arbitrary. Given t_k, s_k, x_k, fix a number $p_k > |t_k|$ and choose s_{k+1} such that $|s_{k+1}| = p_k$, $s_{k+1} > s_k$, $x_{k+1} = \{t \in x_k : t > s_{k+1}\}$ is infinite,[3] and $t_{k+1} \in x_{k+1}$. The set $\{t_k : k < \omega\} \subset x$ obtained in this manner may still not be of any type τ but it is very close. Consider $r_k = t_k \wedge t_{k+1}$ which lie in a chain as $r_k \leq s_{k+1}$. By passing to a subsequence we may suppose that $\max(r_{k+1} \setminus r_k)$ is the same for all $k \in \omega$, and by passing to a further subsequence we may suppose that the pairs $(r_{k+1} \setminus r_k, t_k \setminus r_k)$ are all rungs of the same type τ, and then we will get that $\{t_k : k < \omega\}$ is indeed of type τ. □

4. Finding standard objects

A family of sets I is said to be countably generated in a family J if there exists a countable subset $J_0 \subset J$ such that for every $x \in I$ there exists $y \in J_0$ such that $x \subset y$. The following is restatement of [19, Theorem 3]:

[3] The property that $\{t \in x_k : t > s_k\}$ is infinite is assumed inductively on k.

Theorem **4.1.** — *If* $\{\Gamma_0,\Gamma_1\}$ *are preideals on* $\mathbb{N}$ *such that* Γ_1 *is analytic and is not countably generated in* $\Gamma_0^\perp$, *then there exists an injective function* $u : 2^{<\omega} \longrightarrow \mathbb{N}$ *such that* $u(x)\in\Gamma_i$ *whenever* x *is an* $[i]$*-chain,* $i=0,1$.

Proof. — The actual statement of [19, Theorem 3] says that there is a Γ_0-tree all of whose branches are in Γ_1, which means that there is a family Σ of finite subsets of $\mathbb{N}$ such that

(1) $\emptyset\in\Sigma$,
(2) $\Sigma_a=\{k\in\mathbb{N} : a\cup\{k\}\in\Sigma\}$ is an infinite set in Γ_0, for every $a\in\Sigma$,
(3) if $a_0,a_1,\ldots\in\Sigma$ with $a_0\subset a_1\subset\cdots$, then $\bigcup_{i<\omega}a_i\in\Gamma_1$.

We define inductively the function $v : 2^{<\omega}\longrightarrow\mathbb{N}$ together with a function $a : 2^{<\omega}\longrightarrow\Sigma$ such that $v(s)\in\Sigma_{a(s)}$ in the following way: $a(\emptyset)=\emptyset$, $v(\emptyset)$ is some element of $\Sigma_\emptyset$, $a(s^\frown 0)=a(s)$, $a(s^\frown 1)=a(s)\cup\{v(s)\}$, $v(s^\frown 0)$ is an element of $\Sigma_{a(s^\frown 0)}=\Sigma_{a(s)}$ different from all $v(t)$ that have been previously chosen, and finally $v(s^\frown 1)$ is an element of $\Sigma_{a(s^\frown 1)}$ different from all $v(t)$ that have been previously chosen. Then, we have

(1) If $x=\{s_0<s_1<\cdots\}$ is a $[0]$-chain in $2^{<\omega}$, then $a(s_i)=a(s_0)$ for all $i<\omega$, hence $\{v(s_1),v(s_2),\ldots\}\subset\Sigma_{a(s_0)}$ and therefore $v(x)\in\Gamma_0$, since Σ was a Γ_0-tree.
(2) If $x=\{s_0<s_1<\cdots\}$ is a $[1]$-chain, then $a(s_0)\subset a(s_1)\subset\cdots$ and $v(x)=\{v(s_0),v(s_1),\ldots\}\subset\bigcup_{i<\omega}a(s_i)\in\Gamma_1$ since all branches of Σ are in Γ_1.
(3) Finally, v is injective because at each step we take care that $v(t)$ is different from all previously chosen values of v.

□

Theorem **4.2.** — *If* $\Gamma=\{\Gamma_i : i\in n\}$ *are analytic preideals on the set* $\mathbb{N}$ *which are not separated, then there exists a permutation* $\varepsilon : n\longrightarrow n$ *and a one-to-one map* $u : n^{<\omega}\longrightarrow\mathbb{N}$ *such that* $u(x)\in\Gamma_{\varepsilon(i)}$ *whenever* x *is an* $[i]$*-chain,* $i\in n$.

Proof. — We may assume that Γ is an n_*-gap because otherwise the statement of the theorem is trivial. We will prove the theorem by induction on n. At each step, we shall assume that the statement of the theorem holds for smaller n and we shall find a permutation $\varepsilon : n\longrightarrow n$ and a function $v : n^{<\omega}\longrightarrow\omega$ such that $v(x)\in\Gamma_{\varepsilon(i)}$ whenever x is an $[i]$-chain, $i\in n$, but v will not be one-to-one. Instead, v will have the property that for every $s\in n^{<\omega}$, the set $\{v(s^\frown 0^q) : q<\omega\}$ is infinite.

Let us show how to get the one-to-one function u that we are looking for from a function v as above. For this we shall consider a one-to-one $g : n^{<\omega}\longrightarrow n^{<\omega}$ and we will make $u=vg$. The value of $g(s)$ is defined $\prec$-inductively on s: $g(\emptyset)=\emptyset$ and $g(s^\frown i)=g(s)^\frown i^\frown(0,0,\ldots,0)$ where the number of zeros is chosen so that $v(g(s^\frown i))$ is different from all $v(g(t))$ which have been already defined. Notice that $g(x)$ is an $[i]$-chain whenever x is an $[i]$-chain and $u=vg$ is one-to-one and satisfies the statement of the theorem.

Initial case of the induction: $n = 2$. In view of Theorem 4.1, it is enough to check either Γ_1 is not countably generated in $\Gamma_0^\perp$ or Γ_0 is not countably generated in $\Gamma_1^\perp$. So suppose that we had $x_0 \subset x_1 \subset \cdots$ witnessing that Γ_1 is countably generated in $\Gamma_0^\perp$, and $y_0 \subset y_1 \subset \cdots$ witnessing that Γ_0 is countably generated in $\Gamma_1^\perp$. Then, the elements $x = \bigcup_{k<\omega} x_k \setminus y_k$ and $y = \bigcup_{k<\omega} y_k \setminus x_k$ separate Γ_1 and Γ_0. This finishes the proof of the case when $n = 2$.

Inductive step: We assume that the theorem holds for $n-1$ and we construct the function v for n. We say that a family I of sets is covered by a family J if for every $x \in I$ there exists $y \in J$ such that $x \subset y$. We say that a set $a \subset N$ is small if Γ_a is separated. We say that I covers Γ if I covers $\bigcup_i \Gamma_i$.

Claim **A.** — Γ *cannot be covered by countably many small sets.*

Proof of Claim A. — Assume that $\{a_k : k < \omega\}$ is a sequence of small sets that covers Γ. We can suppose that $a_0 \subset a_1 \subset a_2 \subset \cdots$. For every k, since $\Gamma|_{a_k}$ is separated, there exist sets $a_k(i)$, $i \in n$ such that $\bigcap_i a_k(i) = \emptyset$ and $x \subset^* a_k(i)$ whenever $x \in \Gamma_i|_{a_k}$. By choosing these sets inductively on k, we can make sure that $a_k(i) \subset a_{k+1}(i)$ for every k, i. At the end the sets $a(i) = \bigcup_k a_k(i)$ witness that Γ is separated. This contradiction finishes the proof of Claim A. □

By Claim A, we can find $p \in n$ such that Γ_p is not covered by countably many small sets. Without loss of generality we assume that $p = n-1$. If ε is a permutation of p, we say that $a \subset N$ is ε-small if there exists no one-to-one function $u : p^{<\omega} \longrightarrow a$ such that $u(x) \in \Gamma_{\varepsilon(i)}$ whenever x is an $[i]$-chain, $i < p$.

Claim **B.** — *There exists a permutation $\varepsilon : p \longrightarrow p$ such that Γ_p is not covered by countably many ε-small sets.*

Proof of the claim. — Suppose for contradiction, that Γ_p is countably covered by τ-small sets for every permutation $\tau : p \longrightarrow p$. Let A_τ be a countable family of τ-small sets that covers Γ_p. Then the family of all intersections of the form $a = \bigcap_\tau a_\tau$ with $a_\tau \in A_\tau$ is a countable family that also covers Γ_p. Moreover, each such set a is small by the inductive hypothesis, since we cannot find a permutation τ and a one-to-one function $u : p^{<\omega} \longrightarrow a$ such that $u(x) \in \Gamma_{\tau(i)}$ when x is an $[i]$-chain, $i < p$. This contradicts that Γ_p cannot be countably covered by small sets, and finishes the proof of Claim B. □

A tree on the set $N \times \omega$ is a subset $\Upsilon \subset (N \times \omega)^{<\omega}$ such that if $t \in \Upsilon$ and $s < t$ (in the tree-order, meaning that s is an initial segment of t) then $s \in \Upsilon$. A branch of Υ is an infinite sequence $\{(\xi_k, m_k) : k < \omega\}$ in $N \times \omega$ such that $((\xi_k, m_k) : k < k_0) \in \Upsilon$ for each $k_0 < \omega$. The set of all branches of Υ is denoted by $[\Upsilon]$. Remember that trees on $N \times \omega$ characterize analytic families of subsets of N, in the sense that a family I of subsets of N is analytic if and only if there exists a tree Υ on $\mathrm{N}\times\omega$ such that $\mathrm{I}=[\Upsilon]_1$, where

$$[\Upsilon]_1=\Big\{\{\xi_k:k<\omega\}:\exists\big\{(\xi_k,m_k):k<\omega\big\}\in[\Upsilon]\Big\}$$

Since the ideal Γ_p is analytic, we can find a tree Υ such that $\Gamma_p=[\Upsilon]_1$. For $t\in\Upsilon$ let us denote by $\Upsilon_t=\{s\in\Upsilon:s\geq t \text{ or } s\leq t\}$. Let Υ' be the set of all $t\in\Upsilon$ such that $[\Upsilon_t]$ is not countably covered by ε-small sets. Notice that Υ' is a downwards closed subtree of Υ. Also, for each $t\in\Upsilon'$ we have that $[\Upsilon'_t]$ is not countably covered by ε-small sets, since $[\Upsilon'_t]$ is obtained by removing from $[\Upsilon_t]$ countably many sets of the form $[\Upsilon_s]$ which are countably covered by ε-small sets.

We shall define the function $v:n^{<\omega}\longrightarrow\mathrm{N}$ together with a function $z:n^{<\omega}\longrightarrow\Upsilon'$. For $s\in n^{<\omega}$, let $\mathrm{X}_s=\{s^\frown p^\frown r:r\in p^{<\omega}\}$. By induction on s, we shall define $v|_{\mathrm{X}_s}$ and $z(s)$. For formal reasons, we consider an imaginary element ξ such that $\xi^\frown p=\emptyset$. In this way, ξ is the first step of the induction. We choose $z(\xi)=\emptyset$. Since $[\Upsilon']$ is not covered by countably many ε-small sets, in particular $b=\bigcup[\Upsilon']$ is not ε-small, hence we have a one-to-one function $v_\xi:\mathrm{X}_\xi=p^{<\omega}\longrightarrow b$ such that $v(x)\in\Gamma_{\varepsilon(i)}$ whenever x is an $[i]$-chain, $i<p$. We define $v|_{\mathrm{X}_\xi}=v_\xi$. This finishes the initial step of the inductive definition. We shall suppose along the induction that if $s\in\mathrm{X}_t$, then $v(s)\in\bigcup[\Upsilon'_{z(t)}]_1\setminus z_1(t)$, where $z_1(t)$ is the set of first coordinates of $z(t)$: if $z(t)=\{(\xi_k,m_k):k<k_0\}$ then, $z_1(t)=\{\xi_k:k<k_0\}$.

So suppose that we want to define v on X_s and $z(s)$. Then $s\in\mathrm{X}_t$ for some $t<s$, $s=t^\frown p^\frown r$, $r\in p^{<\omega}$. Therefore $v(s)\in\bigcup[\Upsilon'_{z(t)}]_1\setminus z_1(t)$, so there is a branch of $\Upsilon'_{z(t)}$ such that $v(s)$ appears in the first element at some point—higher than the length of $z(t)$—in the branch. We pick $z(s)>z(t)$ to be a node in this branch which is high enough in order that $v(s)$ appears in the first coordinate. Let $b_s=\bigcup[\Upsilon'_{z(s)}]_1\setminus z_1(s)$ which is not ε-small, so we get a one-to-one $v_s:p^{<\omega}\longrightarrow b_s$ such that $v_s(x)\in\Gamma_{\varepsilon(i)}$ whenever x is an $[i]$-chain, $i<p$. For $\tilde{r}=s^\frown p^\frown r\in\mathrm{X}_s$ we define $v(\tilde{r})=v_s(r)$. This finishes the inductive definition of v.

Let us check that v has the properties that we were looking for. If $t\in n^{<\omega}$, then the set $x=\{t^\frown 0^k:k<\omega\}$ is contained in some X_s, so since the function $v|_{\mathrm{X}_s}$, obtained from v_s, was one-to-one it is clear that $v(x)$ is infinite. Suppose that x is an $[i]$-chain with $i<p$. Then $x\subset\mathrm{X}_s$ for some s, and then $v|_{\mathrm{X}_s}$ was given by v_s which was chosen such that $v_s(x)\in\Gamma_{\varepsilon(i)}$ whenever x is an $[i]$-chain, $i<p$. Finally, suppose that x is a $[p]$-chain, so that $x=\{s_0,s_1,s_2,\ldots\}$ with $s_k{}^\frown p\leq s_{k+1}$ for every $\mathbf{k}<\omega$. Then, by enlarging x intercalating extra elements if necessary we can suppose that $s_{k+1}\in\mathrm{X}_{s_k}$ for every $k<\omega$. Then, by the way that we chose $z(s)$ inductively, we have that $z(s_0)<z(s_1)<\cdots$ and $v(s_k)$ is the first coordinate of a node of $z(s_k)$ above the length of $z(s_{k-1})$. It follows that $\{v(s_k):k<\omega\}\in[\Upsilon']_1\subset[\Upsilon]_1=\Gamma_p$. □

Lemma **4.3.** — *Let Δ_i be the set of all $[i]$-chains of $n^{<\omega}$. Then $\Delta=\{\Delta_i:i\in n\}$ is an n-gap.*

Proof. — The intersection of an $[i]$-chain and a $[j]$-chain contains at most one point when $i\neq j$, so it is clear that the preideals are mutually orthogonal. Let us show that they are not separated. So suppose that we had $a_i\subset n^{<\omega}$ such that $x\subset^* a_i$ for every $[i]$-chain x.

Claim **A.** — *For every $i \in n$ and for every $s \in n^{<\omega}$, there exists $t = t(i, s) \in W_i$ such that $s^\frown t^\frown r \in a_i$ for all $r \in W_i$.*

Proof of the claim. — If not, we would have $i \in n$ and $s \in n^{<\omega}$ such that for every $t \in W_i$ there exists $r \in W_i$ with $s^\frown t^\frown r \notin a_i$. But then we can construct by induction a sequence $\{r_p : p < \omega\} \subset W_i$ such that $r'_p = s^\frown r_0^\frown r_1^\frown \cdots ^\frown r_p \notin a_i$ for every p. But this is a contradiction, because $\{r'_p : p < \omega\}$ is an $[i]$-chain, and we supposed that $x \subset^* a_i$ for all $[i]$-chains. □

Using Claim A, define $s_n = \emptyset$, and by backwards induction $s_i = s_{i+1}{}^\frown t(i, s_{i+1})^\frown i$ for $i = n-1, n-2, \ldots, 0$. In this way $s_n < s_{n-1} < \cdots < s_0$ and $s_i^\frown r \in a_i$ whenever $r \in (i+1)^{<\omega}$. At the end, we have that $s_0^\frown r \in \bigcap_{j\in n} a_j$ for all $r \in W_0$. This shows that $\bigcap_{j\in n} a_j \neq^* \emptyset$. □

Theorem 4.2 is saying that every analytic n_*-gap Γ contains—in a sense—a permutation Δ^ε of the gap Δ in Lemma 4.3, but it is not saying that $\Delta^\varepsilon \leq \Gamma$ because the definition of the order $\leq$ between gaps is much more demanding, as it requires the one-to-one function to respect the orthogonals as well as each of the preideals. If we want to get $\Gamma' \leq \Gamma$, we must allow the rest of types to play, not just the simple types $[i]$, and for this we shall need the machinery of Section 2.

Given a set of types S in $m^{<\omega}$ we denote by Γ_S the preideal of all subsets of $m^{<\omega}$ which are of type τ for some type $\tau \in S$. If $S \cap S' = \emptyset$, then Γ_S and $\Gamma_{S'}$ are orthogonal; indeed, if x and y have different types, then $|x \cap y| \leq 2$.

Corollary **4.4.** — *If $\{S_i : i \in n\}$ are nonempty sets of types in $n^{<\omega}$ with $\bigcap_{i\in n} S_i = \emptyset$, and there is some permutation $\varepsilon : n \longrightarrow n$ such that $[i] \in S_{\varepsilon(i)}$ for every i, then $\Gamma = \{\Gamma_{S_i} : i \in n\}$ is an n_*-gap in $n^{<\omega}$. If the sets S_i are pairwise disjoint, then Γ is an n-gap.*

The existence of the permutation ε is not really necessary for Corollary 4.4 to hold, but the proof is more involved and we shall not include it here. A gap of the form $\{\Gamma_{S_i} : i \in n\}$ as in Corollary 4.4 above will be called a standard n_*-gap. When we have an n_*-gap of the form $\Gamma = \{\Gamma_{S_i} : i \in n\}$ and a type τ, we may, in abuse of notation, write $\tau \in \Gamma_{S_i}$ meaning that $\tau \in S_i$.

Theorem **4.5.** — *For every analytic n_*-gap Γ there exists a standard n_*-gap Γ' such that $\Gamma' \leq \Gamma$.*

Proof. — First, we obtain $u : n^{<\omega} \longrightarrow \omega$ as in Theorem 4.2. Now fix a type τ and we color the sets of type τ into 2^n many colors by declaring that a set x of type τ has color $\xi \subset n$ if $u(x) \in \bigcup_{i\in\xi} \Gamma_i \setminus \bigcup_{i\notin\xi} \Gamma_i$. This coloring is Suslin-measurable since the ideals Γ_i are analytic, so by Theorem 2.5, by passing to a nice subtree we can suppose that all sets of type τ have the same color. We do this for every type τ. For $i \in n$, let S_i be the set of types for which we got that all sets of type τ have color a color ξ with $i \in \xi$ (notice that $[i] \in S_i$). Let us check that under these hypotheses, u witnesses that $\{\Gamma_{S_i} : i \in n\} \leq \Gamma$. It is clear that if $x \in \Gamma_{S_i}$ then $u(x) \in \Gamma_i$. Now, take $x \in \Gamma_{S_i}^\perp$ and let us suppose for contradiction that $u(x) \not\in \Gamma_i^\perp$, so that there exists an infinite $y \subset u(x)$ such that $y \in \Gamma_i$. By Lemma 3.4, we can find an infinite $z \subset u^{-1}(y)$ of some type τ. Since $z \subset x \in \Gamma_{S_i}^\perp$, we must have $\tau \not\in S_i$. But this means that all subsets of type τ had color $\xi \not\ni i$, which implies that $u(z) \not\in \Gamma_i$ which contradicts that $u(z) \subset y \in \Gamma_i$. □

The existence of a finite basis stated in Theorem 1.6 is a corollary of Theorem 4.5 above. There are only finitely many standard n_*-gaps, so if we pick from them those which are minimal among them, that is the finite list of minimal analytic n_*-gaps that lie below any analytic n_*-gap in the order $\leq$.

5. Projective gaps under determinacy

Theorem 5.1 below states that Theorem 4.2 holds not only for analytic gaps, but also for gaps of higher complexity, when assuming determinacy axioms. Theorem 5.1 together with Theorem 2.10 imply that the whole theory developed in this paper holds true for *projective* instead of *analytic* gaps if one assumes Projective Determinacy. The proof that we provide of Theorem 5.1 consists in a reduction to the analytic case of Theorem 4.2.

Theorem **5.1** *(Projective Determinacy). — If $\Gamma = \{\Gamma_i : i \in n\}$ are projective preideals on the set* N *which are not separated, then there exists a permutation $\varepsilon : n \longrightarrow n$ and a one-to-one map $u : n^{<\omega} \longrightarrow \mathbb{N}$ such that $u(x) \in \Gamma_{\varepsilon(i)}$ whenever x is an $[i]$-chain, $i \in n$.*

Proof. — Consider a game $\mathcal{G}(\Gamma)$. Player I plays elements $d_0, d_1, d_2, \dots$ from N in such a way that $d_i \not\in \{d_j : j < i\}$, and Player II responds with $p_0, p_1, p_2, \dots$ from n. At the end, we consider $p_\infty = \limsup_i p_i$ and $i_\infty = \min\{i : \forall j \geq i \ p_j \leq p_\infty\}$. Player I wins if and only if

$$\{d_i : i \geq i_\infty, p_i = p_\infty\} \in \Gamma_{p_\infty}$$

As far as the families Γ_p are projective, this is a projective game, hence determined. It is straightforward to check that Player I having a winning strategy means that there exists a one-to-one map $u : n^{<\omega} \longrightarrow \mathbb{N}$ such that $u(x) \in \Gamma_p$ whenever x is a $[p]$-chain. (The strategy immediately gives a function u which may not be one-to-one, but it is easy to make it injective by restricting to a nice subtree.)

Claim **A.** — *If Player II has a winning strategy in the game $\mathcal{G}(\Gamma)$, then there exist Borel preideals $\tilde{\Gamma}_i \supset \Gamma_i$, such that Player II still has a winning strategy in the game $\mathcal{G}(\tilde{\Gamma})$.*

Proof of Claim A. — Let S be a winning strategy for Player II in the game $\mathcal{G}(\Gamma)$.

For $k < n$ and for $\zeta \in \mathrm{N}^{<\omega}$, we define the set $\mathrm{V}^k(\zeta) \subset \mathrm{N}^{<\omega}$ as the family of all $\zeta^\frown\eta = (\zeta_0, \dots, \zeta_m, \eta_0, \dots, \eta_l)$ such that if

$$\begin{array}{l|cccccc} \textit{Player I} & \zeta_0 & \cdots & \zeta_m & \eta_0 & \cdots & \eta_l \\ \textit{Player II} & p_0 & \cdots & p_m & q_0 & \cdots & q_l \end{array}$$

is played according to the strategy S, then $q_0, \dots q_l \leq k$ and $q_l = k$. We make the convention that $\zeta \in \mathrm{V}^k(\zeta)$.

For every $k < n$ and $\zeta \in \mathrm{N}^{<\omega}$, we also define

$$x^k_\zeta = \left\{d \in \mathrm{N} : \xi_m \neq d \text{ for all } \xi = (\xi_0, \dots, \xi_m) \in \mathrm{V}^k(\zeta) \setminus \{\zeta\}\right\}$$

For every $k < n$, we also define Υ^k to be the set of all $\zeta = (\zeta_0, \dots, \zeta_m) \in \mathrm{N}^{<\omega}$ such that if

$$\begin{array}{l|ccc} \textit{Player I} & \zeta_0 & \cdots & \zeta_m \\ \textit{Player II} & p_0 & \cdots & p_m \end{array}$$

is played according to the strategy S, then there exists no $j \in \{0, \dots, m\}$ such that $p_j = k$ and $p_i < k$ for all $i > j$. □

Claim **A1.** — *For every $k < n$, for every $\zeta \in \Upsilon^k$ and for every $a \in \Gamma_k$, there exists $\xi \in \mathrm{V}^k(\zeta)$ such that $a \subset x^k_\xi$.*

Proof of Claim A1. — Fix $k < n$, $\zeta \in \Upsilon^k$ and $a \in \Gamma_k$ for which the statement of Claim A1 fails. Then, it is possible to construct inductively an infinite set $\{d_1, d_2, d_3, \dots\} \subset a$ together with elements $\eta_1, \eta_2, \dots \in \mathrm{N}^{<\omega}$ such that

$$\xi^m = \zeta^\frown\eta_1^\frown d_1^\frown\eta_2^\frown d_2^\frown \cdots ^\frown \eta_m^\frown d_m \in \mathrm{V}^k(\zeta)$$

for all m. Consider the full infinite round of the game $\mathcal{G}(\Gamma)$, in which Player I moves $\zeta \prec \xi_1 \prec \xi_2 \prec \cdots$ and Player II plays according to the strategy S. In this case, $p_\infty = k$, and the fact that Player II wins means exactly that $\{d_1, d_2, \dots\} \not\in \Gamma_k$. This contradicts that $\{d_1, d_2, \dots\} \subset a \in \Gamma_k$. This finishes the proof of Claim A1. □

For each $k < \omega$, let $\tilde{\Gamma}_k$ be the family of all sets a that satisfy Claim A1. That is,

$$\tilde{\Gamma}_k = \left\{a \subset \mathrm{N} : \forall k < n\, \forall \zeta \in \Upsilon^k\, \exists \xi \in \mathrm{V}^k(\zeta) : a \subset x^k_\xi\right\}$$

This is a Borel preideal, and by Claim A1, $\Gamma_k \subset \tilde{\Gamma}_k$. Now we show that for $\tilde{\Gamma} = \{\tilde{\Gamma}_k : k < n\}$, we can find a winning strategy $\tilde{\mathrm{S}}$ for Player II in the game $\mathcal{G}(\tilde{\Gamma})$. In order to describe the strategy $\tilde{\mathrm{S}}$, let us suppose that Player I plays $d_0, d_1, d_2, \dots$ and we will describe how

Player II must respond. At each move $i < \omega$, we will not only define the number p_i that Player II must play, but also auxiliary numbers $\nu_i^{(k)} < \omega$ for $k < n$.

Player I	d_0	d_1	$\cdots$
Player II	p_0	p_1	$\cdots$
	$\nu_0^{(n-1)}$	$\nu_1^{(n-1)}$	$\cdots$
	$\nu_0^{(n-2)}$	$\nu_1^{(n-2)}$	$\cdots$
	$\cdots$	$\cdots$	$\cdots$
	$\nu_0^{(0)}$	$\nu_1^{(0)}$	$\cdots$

For every $k < n$ and every $\zeta \in \Upsilon^k$, let $\{\xi_\nu^k(\zeta) : \nu < \omega\}$ be an enumeration[4] of $\mathrm{V}^k(\zeta)$. Together with the integers $\nu_i^{(k)}$ we also keep track of elements $\xi_i^{(k)} \in \mathrm{N}^{<\omega}$ defined as follows:

$$\xi_i^{(n-1)} = \xi^{n-1}_{\nu_i^{(n-1)}}(\emptyset) \in \mathrm{V}^{n-1}(\emptyset)$$

$$\xi_i^{(k)} = \xi^k_{\nu_i^{(k)}}\big(\xi_i^{(k+1)}\big) \in \mathrm{V}^k\big(\xi_i^{(k+1)}\big)$$

Notice the following general fact:

Claim **A2.** — *Let* $\{\xi^{(k)} : k < n\} \subset \mathrm{N}^{<\omega}$ *be such that* $\xi^{(n-1)} \in \mathrm{V}^{n-1}(\emptyset)$ *and* $\xi^{(k)} \in \mathrm{V}^k(\xi^{(k+1)})$ *for* $k < n-1$. *Then* $\bigcap_{k<n} x^k_{\xi^{(k)}} = \emptyset$.

Proof of Claim A2. — Suppose for contradiction that $d \in \bigcap_{k<n} x^k_{\xi^{(k)}} = \emptyset$. Consider the finite run of the game $\mathcal{G}(\Gamma)$ played according to strategy S, in which Player I plays the finite sequence $\xi^{(0)\frown} d$. Suppose that Player II responds to the last move d with $p < n$. This would violate that $d \in x^p_{\xi^{(p)}}$ and we get a contradiction. This finishes the proof of Claim A2. □

The initial input is that $\nu_0^{(k)} = 0$ for all $k < n$. Suppose that we are at stage i, that we are given $\nu_i^{(k)}$ (hence also $\xi_i^{(k)}$) for $k < n$, and we describe how Player II must choose p_i and the auxiliary numbers $\nu_{i+1}^{(k)}$.

By Claim A2,

$$\bigcap_{k<n} x^k_{\xi_i^{(k)}} = \emptyset$$

so we can choose

$$p_i = \max\left\{k < n : d_i \notin x^k_{\xi_i^{(k)}}\right\}$$

[4] Notice that $\mathrm{V}^k(\zeta)$ is nonempty as $\zeta \in \mathrm{V}^k(\zeta)$. In case it was finite, an enumeration with repetitions is allowed.

and then

$$\begin{aligned} \nu_{i+1}^{(k)} &= \nu_i^{(k)} && \text{if } k > p_i \\ \nu_{i+1}^{(p_i)} &= \nu_i^{(p_i)} + 1 \\ \nu_{i+1}^{(k)} &= 0 && \text{if } k < p_i \end{aligned}$$

Let us check that this is a winning strategy for Player II in the game $\mathcal{G}(\tilde{\Gamma})$. So consider

$$\begin{array}{l|lll} \textit{Player I} & d_0 & d_1 & \cdots \\ \textit{Player II} & p_0 & p_1 & \cdots \\ \hline & \nu_0^{(n-1)} & \nu_1^{(n-1)} & \cdots \\ & \nu_0^{(n-2)} & \nu_1^{(n-2)} & \cdots \\ & \cdots & \cdots & \cdots \\ & \nu_0^{(0)} & \nu_1^{(0)} & \cdots \end{array}$$

a full infinite run of the game, played according to the strategy $\tilde{S}$, and with the auxiliary $\nu_i^{(k)}$ and $\xi_i^{(k)}$ obtained along the run. The first observation is that

$$p_\infty = \max\left\{k < n : \text{ the sequence } \left\{\nu_i^{(k)} : i < \omega\right\} \text{ is not eventually constant}\right\}$$

$$i_\infty = \min\left\{i < \omega : \forall k > p_\infty \text{ the sequence } \left\{\nu_j^{(k)} : j \geq i\right\} \text{ is constant}\right\}$$

Let ζ be the value at which the sequence $\{\xi_j^{(p_\infty+1)} : j < \omega\}$ stabilizes. If $p_\infty = n-1$, we define $\zeta = \emptyset$. Notice that $\zeta \in \Upsilon^{p_\infty}$ because, in the strategy S, if Player I plays ζ, the last move of Player II is $p_\infty + 1$. We have to check that

$$a = \{d_i : i \geq i_\infty, p_i = p_\infty\} \notin \tilde{\Gamma}_{p_\infty}$$

Assume for contradiction that $a \in \tilde{\Gamma}_{p_\infty}$. By the definition of the $\tilde{\Gamma}_k$'s, there must exist $\xi \in V^{p_\infty}(\zeta)$ such that $a \subset x_\xi^{p_\infty}$. This must appear somewhere in the enumeration that we made, so there exists $\nu < \omega$ such that $\xi = \xi_\nu^{p_\infty}(\zeta)$. By the way in which the p_i and the $\nu_i^{(k)}$ are inductively defined, we have that

$$\left\{\nu_i^{(p_\infty)} : i \geq i_\infty, p_i = p_\infty\right\} = \{0, 1, 2, 3, \ldots\}$$

But on the other hand, since $a \subset x_\xi^{p_\infty}$, we have that $d_i \in x_{\xi_\nu^{p_\infty}(\zeta)}^{p_\infty}$ whenever $i \geq i_\infty$, $p_i = p_\infty$, and the definition of p_i and the other numbers then implies $\nu_i^{(p_\infty)} \neq \nu + 1$ for all $i \geq i_\infty$. This is a contradiction, and it finishes the proof of Claim A.

We come back to the proof of the theorem. For every permutation $\sigma : n \longrightarrow n$ we consider $\Gamma^\sigma = \{\Gamma_{\sigma(i)} : i < n\}$. If there is a permutation σ such that Player I has a winning strategy in the game $\mathcal{G}(\Gamma^\sigma)$ we are done. Otherwise, Player II has a winning strategy in $\mathcal{G}(\Gamma^\sigma)$ for every σ. Consider $\tilde{\Gamma}^\sigma$ given by Claim A, and let

$$\Delta_i = \bigcap_\sigma \tilde{\Gamma}^\sigma_{\sigma^{-1}(i)}$$

and we consider again all the permutations $\Delta^\sigma = \{\Delta_{\sigma(i)} : i < n\}$. For every permutation σ, since $\Delta^\sigma_i \subset \tilde{\Gamma}^\sigma_i$, and Player II has a winning strategy in $\mathcal{G}(\tilde{\Gamma}^\sigma)$, we conclude that Player II has a winning strategy in $\mathcal{G}(\Delta^\sigma)$ as well. In particular, Player I does not have a winning strategy, so there is no one-to-one map $u : n^{<\omega} \longrightarrow \mathbb{N}$ such that $u(x) \in \Delta^\sigma_i$ whenever x is an $[i]$-chain. But the sets Δ_i are Borel, so we can apply Theorem 4.2, and we conclude that Δ is separated. Since $\Gamma_i \subset \Delta_i$, we get that Γ is separated as well. $\square$

The proof of Theorem 5.1 contains implicitly an *asymmetric* version of the theorem, in which on one side no permutation is considered, and on the other side separation is substituted by a winning strategy of Player II. In principle, Player II having a winning strategy gives a coanalytic condition, and the proof above is essentially devoted to transform it into a Borel condition. We found it too technical to state this asymmetric version as a theorem, as it would not look more friendly than referring to the proof of Theorem 5.1.

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