DETERMINANTAL PROBABILITY MEASURES

by Russell LYONS*

ABSTRACT

Determinantal point processes have arisen in diverse settings in recent years and have been investigated intensively. We study basic combinatorial and probabilistic aspects in the discrete case. Our main results concern relationships with matroids, stochastic domination, negative association, completeness for infinite matroids, tail triviality, and a method for extension of results from orthogonal projections to positive contractions. We also present several new avenues for further investigation, involving Hilbert spaces, combinatorics, homology, and group representations, among other areas.

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1. Introduction

A determinantal probability measure is one whose elementary cylinder probabilities are given by determinants. More specifically, suppose that E is a finite or countable set and that Q is an $E \times E$ matrix. For a subset $A \subseteq E$, let $Q{\restriction}A$ denote the submatrix of Q whose rows and columns are indexed by A. If $\mathfrak{S}$ is a random subset of E with the property that for all finite $A \subseteq E$, we have

$$(\mathbf{1.1}) \qquad \mathbf{P}[A \subseteq \mathfrak{S}] = \det(Q{\restriction}A)\,,$$

then we call $\mathbf{P}$ a **determinantal probability measure**. For a trivial example, if $\mathbf{P}$ is the Bernoulli(p) process on E, then (1.1) holds with $Q = pI$, where $0 \le p \le 1$ and

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I is the identity matrix. Slightly more generally, when **P** is a product measure, there is a diagonal matrix Q that makes (1.1) true.

The inclusion-exclusion principle in combination with (1.1) determines the probability of each elementary cylinder event (as a determinant, in fact; see Remark 8.4). Therefore, for every Q, there is at most one probability measure satisfying (1.1). Conversely, it is known (see, e.g., Sect. 8) that there is a determinantal probability measure corresponding to Q if Q is the matrix of a positive contraction on $\ell^2(E)$.

The continuous analogue of determinantal probability measures, i.e., determinantal point processes in $\mathbf{R}^n$, have a long history. This began in the 1950s with Wigner's investigation of the distribution of eigenvalues of certain large random matrices in order to study energy levels of large atoms. The study of eigenvalues of random matrices continues to be an important topic in physics. Dyson (1962) proved that the so-called correlation functions of the distributions that Wigner considered could be described by simple determinants. In the early 1970s, a conversation with Dyson led Montgomery to realize that conjectures he was then formulating concerning the zeroes of the Riemann zeta function were related to the distribution of eigenvalues of random matrices. This idea has been extremely fruitful; see, e.g., the recent reviews by Conrey (2003) and Diaconis (2003). At about that same time, in studying fermions in physics, general point processes with a determinantal form were introduced by Macchi (1975) (see also the references therein) and are known as **fermionic processes** in mathematical physics. The discrete case, which is the one studied here, first appeared in Exercises 5.4.7–5.4.8 of the book by Daley and Vere-Jones (1988), where it is noted that the continuous point processes can be obtained as scaling limits of the discrete measures. Since the end of the 20th century, both the continuous and discrete measures have received much attention, especially, specific measures and, if discrete, their continuous scaling limits. For example, scaling limits of discrete instances have been studied in Borodin (2000), Borodin and Olshanski (2000, 2001, 2002), Borodin, Okounkov, and Olshanski (2000), Johansson (2001, 2002), Okounkov (2001), and Okounkov and Reshetikhin (2001). The literature on the purely continuous case is too voluminous to list here. A general study of the discrete and continuous cases has been undertaken independently by Shirai and Takahashi (2002, 2003) (announced in Shirai and Takahashi (2000)) and Shirai and Yoo (2002), but there is little overlap with our work here. Several aspects of general stationary determinantal probability measures on $\mathbf{Z}^d$ are studied by Lyons and Steif (2003). For a survey of both the discrete and continuous cases, see Soshnikov (2000a).

Our purpose is to establish some new basic combinatorial and probabilistic properties of all (discrete) determinantal probability measures. However, for the benefit of the reader who has not seen any before, we first display a few examples of such measures. Most examples in the literature are too involved even to state here. We restrict our examples to a few that can be detailed easily.

Example **1.1.** — The most well-known example of a (nontrivial discrete) determinantal probability measure is that where $\mathfrak{S}$ is a uniformly chosen random spanning tree of a finite connected graph $G = (\mathsf{V}, \mathsf{E})$ with $E := \mathsf{E}$. In this case, Q is the **transfer current matrix** Y, which is defined as follows. Orient the edges of G arbitrarily. Regard G as an electrical network with each edge having unit conductance. Then $Y(e, f)$ is the amount of current flowing along the edge f when a battery is hooked up between the endpoints of e of such voltage that in the network as a whole, unit current flows from the tail of e to the head of e. The fact that (1.1) holds for the uniform spanning tree is due to Burton and Pemantle (1993) and is called the Transfer Current Theorem. The case with $|A| = 1$ was shown much earlier by Kirchhoff (1847), while the case with $|A| = 2$ was first shown by Brooks, Smith, Stone, and Tutte (1940).

Example **1.2.** — Let $0 < a < 1$ and define the Toeplitz matrix

$$R(i, j) := \frac{1-a}{1+a} a^{|j-i|}$$

for $i, j \in \mathbf{Z}$. Then $\mathbf{P}^R$ is a renewal process on $\mathbf{Z}$ (Soshnikov, 2000a) with renewals at each point of $\mathfrak{S}$. The distance between successive points of $\mathfrak{S}$ has the same distribution as one more than the number of tails until 2 heads appear for a coin that has probability a of coming up tails. Lyons and Steif (2003) extend this example to other regenerative processes.

Example **1.3.** — For $\theta > 0$, consider the probability measure on the set of all partitions λ of all nonnegative integers

$$M^\theta(\lambda) := e^{-\theta} \theta^{|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 .$$

Here, $|\lambda|$ is the sum of the entries of λ and $\dim \lambda$ is the dimension of the representation of the symmetric group on $|\lambda|$ letters that is determined by λ. This measure is a Poissonized version of the Plancherel measure. In order to exhibit M^θ as a determinantal probability measure, as established by Borodin, Okounkov, and Olshanski (2000), we first represent partitions as subsets of $\mathbf{Z} + \frac{1}{2}$ as follows. Consider a partition λ as a Young diagram. Call d the number of squares on the diagonal of λ. The Frobenius coordinates of λ are $\{p_1, \ldots, p_d, q_1, \ldots, q_d\}$, where p_i is the number of squares in the ith row to the right of the diagonal and q_i is number of squares in the ith column below the diagonal. Following Vershik and Kerov (1981), define the **modified Frobenius coordinates** $\mathrm{Fr}(\lambda)$ of λ by

$$\mathrm{Fr}(\lambda) := \left\{ p_1 + \frac{1}{2}, \ldots, p_d + \frac{1}{2}, -q_1 - \frac{1}{2}, \ldots, -q_d - \frac{1}{2} \right\} \subset \mathbf{Z} + \frac{1}{2} .$$

Clearly, the map $\lambda \mapsto \mathrm{Fr}(\lambda)$ is injective. Then the law of $\mathrm{Fr}(\lambda)$ is $\mathbf{P}^{\mathsf{K}}$ when λ has the law M^θ, where the matrix K is

$$\mathsf{K}(x,y) := \begin{cases} \sqrt{\theta}\, \dfrac{\mathsf{k}_+(|x|,|y|)}{|x|-|y|} & \text{if } xy > 0, \\ \sqrt{\theta}\, \dfrac{\mathsf{k}_-(|x|,|y|)}{x-y} & \text{if } xy < 0, \end{cases}$$

with the functions $\mathsf{k}_\pm$ defined by

$$\mathsf{k}_+(x,y) := \mathrm{J}_{x-\frac{1}{2}}\mathrm{J}_{y+\frac{1}{2}} - \mathrm{J}_{x+\frac{1}{2}}\mathrm{J}_{y-\frac{1}{2}}\,,$$
$$\mathsf{k}_-(x,y) := \mathrm{J}_{x-\frac{1}{2}}\mathrm{J}_{y-\frac{1}{2}} + \mathrm{J}_{x+\frac{1}{2}}\mathrm{J}_{y+\frac{1}{2}}\,,$$

where $\mathrm{J}_x := \mathrm{J}_x(2\sqrt{\theta})$ is the Bessel function of order x and argument $2\sqrt{\theta}$. Although K is not self-adjoint on $\ell^2(\mathbf{Z}+\frac{1}{2})$, its restrictions to $\ell^2(\mathbf{Z}^+ - \frac{1}{2})$ and to $\ell^2(\mathbf{Z}^- + \frac{1}{2})$ are.

Our main results are as follows (see Theorem 8.1). We show that for any positive contraction, Q, the measure $\mathbf{P}^{\mathrm{Q}}$ has a very strong negative association property called "conditional negative associations with external fields". Also, if Q_1 and Q_2 are commuting positive contractions and $\mathrm{Q}_1 \le \mathrm{Q}_2$, then $\mathbf{P}^{\mathrm{Q}_1}$ is stochastically dominated by $\mathbf{P}^{\mathrm{Q}_2}$. These properties, especially the former, are powerful tools, comparable to the well-known FKG inequalities representing positive association that hold for many models of statistical mechanics. For example, these results are crucial to most of the results in Lyons and Steif (2003). We also show that $\mathbf{P}^{\mathrm{Q}}$ has a trivial (full) tail σ-field for all Q; this was shown independently by Shirai and Takahashi (2003) when the spectrum of Q lies in (0, 1). (Ironically, our proof is based on the case when the spectrum of Q equals {0, 1}; see below.) Another main result, Theorem 7.2, is perhaps best appreciated at this point by giving two of its consequences. The first consequence is a theorem of Morris (2003), which says that on any graph whatsoever, a.s. each tree in the wired uniform spanning forest is recurrent for simple random walk (see Sect. 3 for definitions). A second corollary is the following. Let $\mathbf{T} := \mathbf{R}/\mathbf{Z}$ be the unit circle equipped with unit Lebesgue measure. For a measurable function $f : \mathbf{T} \to \mathbf{C}$ and an integer n, the **Fourier coefficient** of f at n is

$$\widehat{f}(n) := \int_{\mathbf{T}} f(t) e^{-2\pi i n t}\, dt\,.$$

For the meaning of "Beurling-Malliavin density", see Definition 7.13.

Corollary **1.4.** — *Let* $\mathrm{A} \subset \mathbf{T}$ *be Lebesgue measurable with measure* $|\mathrm{A}|$. *Then there is a set* S *of Beurling-Malliavin density* $|\mathrm{A}|$ *in* $\mathbf{Z}$ *such that if* $f \in \mathrm{L}^2(\mathbf{T})$ *vanishes a.e. on* A *and* $\widehat{f}$ *vanishes off* S, *then* $f = 0$ *a.e. Indeed, let* $\mathbf{P}^{\mathrm{A}}$ *be the determinantal probability measure on* $2^{\mathbf{Z}}$ *corresponding to the Toeplitz matrix* $(j,k) \mapsto \widehat{\mathbf{1}_{\mathrm{A}}}(k-j)$. *Then* $\mathbf{P}^{\mathrm{A}}$*-a.e.* $\mathrm{S} \subset \mathbf{Z}$ *has this property.*

This can be compared to a theorem of Bourgain and Tzafriri (1987), according to which there is a set $S \subset \mathbf{Z}$ of density at least $2^{-8}|A|$ such that if $f \in L^2(\mathbf{T})$ and $\widehat{f}$ vanishes off S, then

$$|A|^{-1} \int_A |f(t)|^2 \, dt \geq 2^{-16} \|f\|_2^2 .$$

Here, "density" is understood in the ordinary sense, but S is found so that the density of S is also equal to its Schnirelman density. It would be interesting to find a common strengthening of Corollary 1.4 and the theorem of Bourgain and Tzafriri. Note that for Corollary 1.4, it is important to use a strong notion of density, as otherwise a subset of $\mathbf{N}$ would be a trivial example for any $|A| \leq 1/2$. (Bourgain and Tzafriri (1987) find $S \subset \mathbf{N}$, but they could just as well have chosen S in $\mathbf{Z}$ with the same density.)

As hinted at above, we shall base all our results on the case where Q is an orthogonal projection. This is accomplished by a new and simple reduction method from positive contractions to orthogonal projections. A main tool in the case of orthogonal projections is a new representation of the associated probability measure via exterior algebra. We are also led naturally in this case to relations with matroids. Finally, we detail a large number of new questions and conjectures that we find quite intriguing. These range over a number of areas of mathematics, including Hilbert spaces, combinatorics, homology, and group representations, *inter alia*.

We now give an outline of the rest of the paper. We begin in Sect. 2 with the relationship of determinantal probability measures to matroids. Matroids provide a useful organizing framework as well as an inspirational viewpoint. Prior research has contributed to a deep understanding of the particular case of the uniform spanning tree measure (Example 1.1) and its extensions to infinite graphs. Since generalizing this understanding is one goal of the present investigations and of several open questions, we provide a quick summary of the relevant facts for uniform spanning trees and forests in Sect. 3. For the sake of probabilists, we review exterior algebra in Sect. 4. We use exterior algebra in Sect. 5 to give the basic properties of determinantal probability measures and in Sect. 6 to prove our stochastic comparison inequalities. The case of infinite dimensions is treated in Sect. 7. In Sect. 8, we explain how to associate probability measures to positive contractions and show how results in this more general setting follow easily from results for the special case of orthogonal projections. We outline many areas of open questions in Sects. 9–13.

2. Matroids

The set of spanning trees of a finite connected graph is not only the best-known ground set for discrete determinantal probability measures, but also the most well-known example of the set of bases in a matroid. Indeed, it was one of the two principal motivating examples for the introduction of the theory of matroids by Whitney (1935). Moreover, matroids are intrinsically linked to determinantal probability measures. To see this, recall the definition of a matroid.

A matroid is a simple combinatorial object satisfying just one axiom (for more background, see Welsh (1976) or Oxley (1992)). Namely, let E be a finite set and let $\mathscr{B}$ be a nonempty collection of subsets of E. We call the pair $\mathscr{M} := (E, \mathscr{B})$ a **matroid** with **bases** $\mathscr{B}$ if the following exchange property is satisfied:

$$\forall B, B' \in \mathscr{B} \quad \forall e \in B \setminus B' \quad \exists e' \in B' \setminus B \quad (B \setminus \{e\}) \cup \{e'\} \in \mathscr{B}\,.$$

All bases have the same cardinality, called the **rank** of the matroid. Two fundamental examples are:

- E is the set of edges of a finite connected graph G and $\mathscr{B}$ is the set of spanning trees of G.
- E is a set of vectors in a vector space and $\mathscr{B}$ is the collection of maximal linearly independent subsets of E, where "maximal" means with respect to inclusion.

The first of these two classes of examples is called **graphical**, while the second is called **vectorial**. The **dual** of a matroid $\mathscr{M} = (E, \mathscr{B})$ is the matroid $\mathscr{M}^\perp := (E, \mathscr{B}^\perp)$, where $\mathscr{B}^\perp := \{E \setminus B\,;\ B \in \mathscr{B}\}$. The dual matroid is also called the **orthogonal** matroid.

Any matroid that is isomorphic to a vectorial matroid is called **representable**. Graphical matroids are **regular**, i.e., representable over every field.

Each representation of a vectorial matroid over a subfield of the complex numbers $\mathbf{C}$ gives rise to a determinantal probability measure in the following fashion. Let $\mathscr{M} = (E, \mathscr{B})$ be a matroid of rank r. If $\mathscr{M}$ can be represented over a field $\mathbf{F}$, then the usual way of specifying such a representation is by an $(s \times E)$-matrix M whose columns are the vectors in $\mathbf{F}^s$ representing $\mathscr{M}$ in the usual basis of $\mathbf{F}^s$. For what follows below, in fact any basis of $\mathbf{F}^s$ may be used. One calls M a **coordinatization matrix** of $\mathscr{M}$. Now the rank of the matrix M is also r. If a basis for the column space is used instead of the usual basis of $\mathbf{F}^s$, then we may take $s = r$. In any case, the row space $H \subseteq \mathbf{F}^E$ of M is r-dimensional and we may assume that the first r rows of M span H.

When $\mathbf{F} \subseteq \mathbf{C}$, a determinantal probability measure $\mathbf{P}^H$ corresponding to the row space H (or indeed to any subspace H of $\ell^2(E;\, \mathbf{F})$) can be defined via any of several equivalent formulae. Conceptually the simplest definition is to use (1.1) with Q being the matrix of the orthogonal projection $P_H : \ell^2(E) \to H$, where the matrix of P_H is defined with respect to the usual basis of $\ell^2(E)$. If, however, the coordinatization matrix M is more available than P_H, then one can proceed as follows.

For an r-element subset B of E, let M_B denote the $(r \times r)$-matrix determined by the first r rows of M and those columns of M indexed by $e \in B$. Let $M_{(r)}$ denote the matrix formed by the *entire* first r rows of M. In Sect. 5, we shall see that

$$\mathbf{P}^H[B] = |\det M_B|^2 / \det(M_{(r)} M_{(r)}^*)\,, \tag{2.1}$$

where the superscript $*$ denotes adjoint. (One way to see that this defines a probability measure is to use the Cauchy-Binet formula.) Identify each $e \in \mathrm{E}$ with the corresponding unit basis vector of $\ell^2(\mathrm{E})$. Equation (2.1) shows that the matrix M is a coordinatization matrix of $\mathscr{M}$ iff the row space H is a **subspace representation** of $\mathscr{M}$ in the sense that for r-element subsets $\mathrm{B} \subseteq \mathrm{E}$, we have $\mathrm{B} \in \mathscr{B}$ iff $\mathrm{P_H B}$ is a basis for H. (We shall also say simply that H **represents** $\mathscr{M}$.)

Now use row operations to transform the first r rows of M to an orthonormal basis for H. Let $\widehat{\mathrm{M}}$ be the $(r \times \mathrm{E})$-matrix formed from these new first r rows. We shall see that

$$\textbf{(2.2)} \qquad \widehat{\mathrm{M}}^*\widehat{\mathrm{M}} \text{ is the transpose of the matrix of } \mathrm{P_H}\,.$$

We now interpret all this for the case of a graphical matroid. Given a finite connected graph $\mathrm{G} = (\mathsf{V}, \mathsf{E})$, choose an orientation for each edge. Given $x \in \mathsf{V}$, define the **star** at x to be the vector $\star_x := \star_x^{\mathrm{G}} := \sum_{e \in \mathsf{E}} a(x, e)e \in \ell^2(\mathsf{E})$, where

$$a(x, e) := \begin{cases} 0 & \text{if } x \text{ is not incident to } e, \\ 1 & \text{if } x \text{ is the tail of } e, \\ -1 & \text{if } x \text{ is the head of } e \end{cases}$$

are the entries of the $\mathsf{V} \times \mathsf{E}$ **vertex-edge incidence matrix**. Let $\bigstar := \bigstar(\mathrm{G})$ be the linear span of the stars, which is the same as the row space of $[a(\bullet, \bullet)]$. The standard coordinatization matrix of the matroid of spanning trees, the graphic matroid of G, is $[a(\bullet, \bullet)]$. It is easy to check that this does represent the graphic matroid. It follows that $\bigstar$ is a subspace representation of the graphic matroid, which is also well known. It is further well known that the matrix of $\mathrm{P_H}$ is the transfer current matrix Y; see, e.g., Benjamini, Lyons, Peres, and Schramm (2001), herein after referred to as BLPS (2001). It is well known that the numerator of the right-hand side of (2.1) is 1 when B is a spanning tree of G and 0 otherwise. It follows that $\mathbf{P}^{\bigstar}$ is the uniform measure on spanning trees of G. It also follows that the denominator of the right-hand side of (2.1) is the number of spanning trees. To see what the matrix $\mathrm{M}_{(r)}\mathrm{M}_{(r)}^*$ is that appears in the denominator of the right-hand side of (2.1), let the vertices of the graph be $v_0, \ldots, v_r$. For $1 \le i, j \le r$ only, let $b_{i,j}$ be -1 if v_i and v_j are adjacent, the degree of v_i if $i = j$, and 0 otherwise. This is the same as the combinatorial Laplacian after the row and column corresponding to v_0 are removed. Then if the first r rows of M correspond to v_i for $1 \le i \le r$, we obtain that $\mathrm{M}_{(r)}\mathrm{M}_{(r)}^*$ is $[b_{i,j}]$. The fact that the determinant of this matrix is the number of spanning trees is known as the Matrix-Tree Theorem.

As we shall see in Corollary 5.5, those matroids $(\mathrm{E}, \mathscr{B})$ having a subspace representation H whose associated probability measure $\mathbf{P}^{\mathrm{H}}$ is the uniform measure on $\mathscr{B}$ are precisely the regular matroids in the case of real scalars and complex unimodular matroids (also called sixth-root-of-unity matroids) in the case of complex scalars.

There is a relationship to oriented matroids that may be worth noting. In case the field of a vectorial matroid is ordered, such as $\mathbf{R}$, there is an additional structure making it an oriented matroid. Rather than define oriented matroids in general, we give a definition in the representable case that is convenient for our purposes: a **real oriented matroid** χ of rank r on $\mathrm{E} = \{1, 2, \ldots, n\}$ is a map $\chi : \mathrm{E}^r \to \{-1, 0, +1\}$ such that for some independent vectors $v_1, \ldots, v_r \in \mathbf{R}^{\mathrm{E}}$,

$$\chi(k_1, \ldots, k_r) = \operatorname{sgn} \det[(v_i,\ e_{k_j})]_{i,j \le r}\,, \tag{2.3}$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbf{R}^{\mathrm{E}}$ and $(\bullet,\ \bullet)$ is the usual inner product in $\mathbf{R}^{\mathrm{E}}$. The sets $\{k_1, \ldots, k_r\}$ with $\chi(k_1, \ldots, k_r) \neq 0$ are the bases of a matroid on E.

Let H be the linear span of the vectors v_k appearing in the preceding definition. Choosing a different basis for H and defining a new χ' in terms of this new basis by (2.3) will lead only to $\chi' = \pm\chi$. Of course, the determinants themselves in (2.3) can change dramatically. However, if $\{v_1, \ldots, v_r\}$ are orthonormal, then the determinants are fixed up to this sign change and they give a determinantal probability measure. Namely, if $\mathfrak{B}$ denotes a random base chosen with probability

$$\mathbf{P}^{\mathrm{H}}[\mathfrak{B} = \{k_1, \ldots, k_r\}] := (\det[(v_i,\ e_{k_j})]_{i,j \le r})^2\,, \tag{2.4}$$

then this agrees with (2.1), as shown in Sect. 5.

3. Uniform spanning forests

The subject of random spanning trees of a graph goes back to Kirchhoff (1847), who discovered several relations to electrical networks, one of them mentioned already in Sect. 1. His work also suggested the Matrix-Tree Theorem.

In recent decades, computer scientists have developed various algorithms for generating spanning trees of a finite graph at random according to the uniform measure. In particular, such algorithms are closely related to generating states at random from any Markov chain. See Propp and Wilson (1998) for more on this issue.

Early algorithms for generating a random spanning tree used the Matrix-Tree Theorem. A much better algorithm than these early ones, especially for probabilists, was introduced by Aldous (1990) and Broder (1989). It says that if you start a simple random walk at *any* vertex of a finite graph G and draw every edge it crosses except when it would complete a cycle (i.e., except when it arrives at a previously-visited vertex), then when no more edges can be added without creating a cycle, what will be drawn is a uniformly chosen spanning tree of G. This beautiful algorithm is quite efficient and useful for theoretical analysis, yet Wilson (1996) found an even better one. It is lengthier to describe, so we forego its description, but it is faster and more flexible.

The study of the analogue on an infinite graph of a uniform spanning tree was begun by Pemantle (1991) at the suggestion of the present author. Pemantle showed that if an infinite connected graph G is exhausted by a sequence of finite connected subgraphs G_n, then the weak limit of the uniform spanning tree measures on G_n exists. However, it may happen that the limit measure is not supported on trees, but on forests. This limit measure is now called the **free uniform spanning forest** on G, denoted FSF. Considerations of electrical networks play the dominant role in the proof of existence of the limit. If G is itself a tree, then this measure is trivial, namely, it is concentrated on {G}. Therefore, Häggström (1995) introduced another limit that had been considered more implicitly by Pemantle (1991) on $\mathbf{Z}^d$, namely, the weak limit of the uniform spanning tree measures on G_n^*, where G_n^* is the graph G_n with its boundary identified to a single vertex. As Pemantle (1991) showed, this limit also always exists on any graph and is now called the **wired uniform spanning forest**, denoted WSF.

In many cases, the free and the wired limits are the same. In particular, this is the case on all euclidean lattices such as $\mathbf{Z}^d$. The general question of when the free and wired uniform spanning forest measures are the same turns out to be quite interesting: The measures are the same iff there are no nonconstant harmonic Dirichlet functions on G (see BLPS (2001)).

Among the notable results that Pemantle (1991) proved was that on $\mathbf{Z}^d$, the uniform spanning forest is a single tree with a single end a.s. if $2 \le d \le 4$; when $d \ge 5$, there are infinitely many trees a.s., each with at most two ends. BLPS (2001) shows that, in fact, not only in $\mathbf{Z}^d$, but on any Cayley graph of a finitely generated group other than a finite extension of $\mathbf{Z}$, each tree in the WSF has but a single end a.s. One of Pemantle's main tools was the Aldous/Broder algorithm, while BLPS (2001) relied on a modification of Wilson's algorithm.

Another result of Pemantle (1991) was that on $\mathbf{Z}^d$, the tail of the uniform spanning forest measure is trivial. This is extended to both the free and the wired spanning forests on all graphs in BLPS (2001).

In the paper BLPS (2001), it was noted that the Transfer Current Theorem extends to the free and wired spanning forests if one uses the free and wired currents, respectively. To explain this, note that the orthocomplement of the row space $\bigstar(G)$ of the vertex-edge incidence matrix of a finite graph G is the kernel, denoted $\diamondsuit(G)$, of the matrix. We call $\bigstar(G)$ the **star space** of G and $\diamondsuit(G)$ the **cycle space** of G. For an infinite graph $G = (\mathsf{V}, \mathsf{E})$ exhausted by finite subgraphs G_n, we let $\bigstar(G)$ be the closure of $\bigcup \bigstar(G_n^*)$ and $\diamondsuit(G)$ be the closure of $\bigcup \diamondsuit(G_n)$, where we take the closure in $\ell^2(\mathsf{E})$. Then WSF is the determinantal probability measure corresponding to the projection $P_{\bigstar(G)}$, while FSF is the determinantal probability measure corresponding to $P_{\diamondsuit(G)}^{\perp} := P_{\diamondsuit(G)^{\perp}}$. In particular, $\mathsf{WSF} = \mathsf{FSF}$ iff $\bigstar(G) = \diamondsuit(G)^{\perp}$.

While the wired spanning forest is quite well understood, the free spanning forest measure is in general poorly understood. A more detailed summary of uniform spanning forest measures can be found in Lyons (1998).

4. Review of exterior algebra

Let E be a finite or countable set. Identify E with the standard orthonormal basis of the real or complex Hilbert space $\ell^2(\mathrm{E})$. For $k \ge 1$, let E_k denote a collection of ordered k-element subsets of E such that each k-element subset of E appears exactly once in E_k in some ordering. Define

$$\Lambda^k \mathrm{E} := \bigwedge^k \ell^2(\mathrm{E}) := \ell^2\Big(\big\{e_1 \wedge \cdots \wedge e_k\,;\ \langle e_1, \ldots, e_k\rangle \in \mathrm{E}_k\big\}\Big)\,.$$

If $k > |\mathrm{E}|$, then $\mathrm{E}_k = \emptyset$ and $\Lambda^k\mathrm{E} = \{0\}$. We also define $\Lambda^0\mathrm{E}$ to be the scalar field, **R** or **C**. The elements of $\Lambda^k\mathrm{E}$ are called **multivectors** of **rank** k, or k**-vectors** for short. We then define the **exterior** (or **wedge**) **product** of multivectors in the usual alternating multilinear way: $\bigwedge_{i=1}^k e_{\sigma(i)} = \mathrm{sgn}(\sigma)\bigwedge_{i=1}^k e_i$ for any permutation σ of $\{1, 2, \ldots, k\}$ and $\bigwedge_{i=1}^k \sum_{e\in\mathrm{E}'} a_i(e)e = \sum_{e_1,\ldots,e_k\in\mathrm{E}'} \prod_{j=1}^k a_j(e_j)\bigwedge_{i=1}^k e_i$ for any scalars $a_i(e)$ ($i \in [1, k]$, $e \in \mathrm{E}'$), and any finite $\mathrm{E}' \subseteq \mathrm{E}$. (Thus, $\bigwedge_{i=1}^k e_i = 0$ unless all e_i are distinct.) The inner product on $\Lambda^k\mathrm{E}$ satisfies

$$(u_1 \wedge \cdots \wedge u_k,\ v_1 \wedge \cdots \wedge v_k) = \det\big[(u_i,\ v_j)\big]_{i,j\in[1,k]} \tag{4.1}$$

when u_i and v_j are 1-vectors. (This also shows that the inner product on $\Lambda^k\mathrm{E}$ does not depend on the choice of orthonormal basis of $\ell^2(\mathrm{E})$.) We then define the **exterior** (or **Grassmann**) **algebra** $\mathrm{Ext}\big(\ell^2(\mathrm{E})\big) := \mathrm{Ext}(\mathrm{E}) := \bigoplus_{k\ge0}\Lambda^k\mathrm{E}$, where the summands are declared orthogonal, making it into a Hilbert space. (Throughout the paper, $\oplus$ is used to indicate the sum of orthogonal summands, or, if there are an infinite number of orthogonal summands, the closure of their sum.) Vectors $u_1, \ldots, u_k \in \ell^2(\mathrm{E})$ are linearly independent iff $u_1 \wedge \cdots \wedge u_k \ne 0$. For a k-element subset $\mathrm{A} \subseteq \mathrm{E}$ with ordering $\langle e_i\rangle$ in E_k, write

$$\theta_\mathrm{A} := \bigwedge_{i=1}^k e_i\,.$$

We also write

$$\bigwedge_{e\in\mathrm{A}} f(e) := \bigwedge_{i=1}^k f(e_i)$$

for any function $f : \mathrm{E} \to \ell^2(\mathrm{E})$.

If H is a (closed) linear subspace of $\ell^2(\mathrm{E})$, then we identify Ext(H) with its inclusion in Ext(E). That is, $\bigwedge^k\mathrm{H}$ is the closure of the linear span of $\{v_1 \wedge \cdots \wedge v_k;\ v_1, \ldots, v_k \in \mathrm{H}\}$. In particular, if $\dim\mathrm{H} = r < \infty$, then $\bigwedge^r\mathrm{H}$ is a 1-dimensional subspace of Ext(E); denote by ξ_H a unit multivector in this subspace, which is unique up to sign in the real case and unique up to a scalar unit-modulus factor in the complex case, i.e., up to **signum**. We denote by P_H the orthogonal projection onto H for any subspace $H \subseteq \ell^2(E)$ or, more generally, $H \subseteq \mathrm{Ext}(E)$.

Lemma **4.1.** — *For any subspace* $H \subseteq \ell^2(E)$, *any* $k \geq 1$, *and any* $u_1, \ldots, u_k \in \ell^2(E)$,

$$P_{\mathrm{Ext}(H)}(u_1 \wedge \cdots \wedge u_k) = (P_H u_1) \wedge \cdots \wedge (P_H u_k) .$$

Proof. — Write

$$u_1 \wedge \cdots \wedge u_k = \big(P_H u_1 + P_H^{\perp} u_1\big) \wedge \cdots \wedge \big(P_H u_k + P_H^{\perp} u_k\big)$$

and expand the product. All terms but $P_H u_1 \wedge \cdots \wedge P_H u_k$ have a factor of $P_H^{\perp} u$ in them, making them orthogonal to Ext(H) by (4.1). □

A multivector is called **simple** or **decomposable** if it is the wedge product of 1-vectors. Whitney (1957), p. 49, shows that

$$(\mathbf{4.2}) \qquad \|\mathbf{u} \wedge \mathbf{v}\| \leq \|\mathbf{u}\| \|\mathbf{v}\| \quad \text{if either } \mathbf{u} \text{ or } \mathbf{v} \text{ is simple.}$$

We shall use the **interior product** defined by duality:

$$(\mathbf{u} \vee \mathbf{v},\ \mathbf{w}) = (\mathbf{u},\ \mathbf{w} \wedge \mathbf{v}) \qquad (\mathbf{u} \in \Lambda^{k+l} E,\ \ \mathbf{v} \in \Lambda^{l} E,\ \ \mathbf{w} \in \Lambda^{k} E) .$$

In particular, if $e \in E$ and $\mathbf{u}$ is a multivector that does not contain any term with e in it (that is, $\mathbf{u} \in \mathrm{Ext}(e^{\perp})$), then $(\mathbf{u} \wedge e) \vee e = \mathbf{u}$ and $\mathbf{u} \vee e = 0$. More generally, if $v \in \ell^2(E)$ with $\|v\| = 1$ and $\mathbf{u} \in \mathrm{Ext}(v^{\perp})$, then $(\mathbf{u} \wedge v) \vee v = \mathbf{u}$ and $\mathbf{u} \vee v = 0$. Note that the interior product is sesquilinear, not bilinear, over $\mathbf{C}$.

For $e \in E$, write $[e]$ for the subspace of scalar multiples of e in $\ell^2(E)$. If H is a finite-dimensional subspace of $\ell^2(E)$ and $e \notin H$, then

$$(\mathbf{4.3}) \qquad \xi_H \wedge e = \left\|P_H^{\perp} e\right\| \xi_{H+[e]}$$

(up to signum). To see this, let $u_1, u_2, \ldots, u_r$ be an orthonormal basis of H, where $r = \dim H$. Put $v := P_H^{\perp} e / \|P_H^{\perp} e\|$. Then $u_1, \ldots, u_r, v$ is an orthonormal basis of $H + [e]$, whence

$$\xi_{H+[e]} = u_1 \wedge u_2 \wedge \cdots \wedge u_r \wedge v = \xi_H \wedge v = \xi_H \wedge e / \left\|P_H^{\perp} e\right\|$$

since $\xi_H \wedge P_H e = 0$. This shows (4.3). Similarly, if $e \notin H^{\perp}$, then

$$(\mathbf{4.4}) \qquad \xi_H \vee e = \|P_H e\| \, \xi_{H \cap e^{\perp}}$$

(up to signum). Indeed, put $w_1 := P_H e / \|P_H e\|$. Let $w_1, w_2, \ldots, w_r$ be an orthonormal basis of H with $\xi_H = w_1 \wedge w_2 \wedge \cdots \wedge w_r$. Then

$$\xi_H \vee e = \xi_H \vee P_H e = (-1)^{r-1} \|P_H e\| \, w_2 \wedge w_3 \cdots \wedge w_r$$

(up to signum), as desired.

Finally, we claim that

$$(\mathbf{4.5}) \qquad \forall u, v \in \ell^2(\mathrm{E}) \qquad \big(\xi_{\mathrm{H}} \vee u,\ \xi_{\mathrm{H}} \vee v\big) = (\mathrm{P}_{\mathrm{H}} v,\ u)\,.$$

Indeed, $\xi_{\mathrm{H}} \vee u = \xi_{\mathrm{H}} \vee \mathrm{P}_{\mathrm{H}} u$, so this is equivalent to

$$\big(\xi_{\mathrm{H}} \vee \mathrm{P}_{\mathrm{H}} u,\ \xi_{\mathrm{H}} \vee \mathrm{P}_{\mathrm{H}} v\big) = (\mathrm{P}_{\mathrm{H}} v,\ \mathrm{P}_{\mathrm{H}} u)\,.$$

Thus, it suffices to show that

$$\forall u, v \in \mathrm{H} \qquad \big(\xi_{\mathrm{H}} \vee u,\ \xi_{\mathrm{H}} \vee v\big) = (v,\ u)\,.$$

By sesquilinearity, it suffices to show this for u, v members of an orthonormal basis of H. But then it is obvious.

For a more detailed presentation of exterior algebra, see Whitney (1957).

5. Probability measures on bases

Any unit vector v in a Hilbert space with orthonormal basis E gives a probability measure $\mathbf{P}^v$ on E, namely, $\mathbf{P}^v[\{e\}] := |(v,\ e)|^2$ for $e \in \mathrm{E}$. Applying this simple idea to multivectors instead, we shall obtain the probability measures $\mathbf{P}^{\mathrm{H}}$ of Sect. 1.

Let H be a subspace representation of the matroid $\mathscr{M} = (\mathrm{E}, \mathscr{B})$ of rank r. Then the non-0 coefficients in ξ_{H} with respect to the standard basis of $\Lambda^r \mathrm{E}$ are exactly those of the multivectors θ_{B} ($\mathrm{B} \in \mathscr{B}$). Indeed, by Lemma 4.1, the coefficient in ξ_{H} of $\theta_{\mathrm{B}} = \bigwedge_{i=1}^r e_i$ satisfies

$$|(\xi_{\mathrm{H}},\ \theta_{\mathrm{B}})| = \|\mathrm{P}_{\mathrm{Ext(H)}}(\bigwedge_{i=1}^{r} e_i)\| = \|\bigwedge_{i} \mathrm{P}_{\mathrm{H}} e_i\|\,,$$

which is non-0 iff $\langle \mathrm{P}_{\mathrm{H}} e_i \rangle$ are linearly independent. Furthermore, we may define the probability measure $\mathbf{P}^{\mathrm{H}}$ on $\mathscr{B}$ by

$$(\mathbf{5.1}) \qquad \mathbf{P}^{\mathrm{H}}[\mathrm{B}] := |\,(\xi_{\mathrm{H}},\ \theta_{\mathrm{B}})\,|^2\,.$$

We may also write (5.1) as

$$\xi_{\mathrm{H}} = \sum_{\mathrm{B} \in \mathscr{B}} \epsilon_{\mathrm{B}} \sqrt{\mathbf{P}^{\mathrm{H}}[\mathrm{B}]}\,\theta_{\mathrm{B}}$$

for some ϵ_{B} of absolute value 1, or alternatively as

$$\mathbf{P}^{\mathrm{H}}[\mathfrak{B} = \mathrm{B}] = \|\mathrm{P}_{\mathrm{Ext(H)}} \theta_{\mathrm{B}}\|^2 = (\mathrm{P}_{\mathrm{Ext(H)}} \theta_{\mathrm{B}},\ \theta_{\mathrm{B}})\,.$$

To see that (5.1) agrees with (2.4), we just use (4.1) and the fact that $\xi_{\mathrm{H}} = c \bigwedge_i v_i$ for some scalar c with $|c| = 1$.

To show that the definition (5.1) satisfies (1.1) for the matrix of P_H, observe that

$$\mathbf{P}^H[\mathfrak{B} = B] = \big(P_{\mathrm{Ext}(H)}\theta_B,\ \theta_B\big) = \Bigg(\bigwedge_{e\in B} P_H e,\ \bigwedge_{e\in B} e\Bigg) = \det[(P_H e, f)]_{e,f\in B}$$

by (4.1). This shows that (1.1) holds for $A \in \mathscr{B}$ since $|\mathfrak{B}| = r$ $\mathbf{P}^H$-a.s. We now prove the general case by proving an extension of it. We shall use the convention that $\theta_\emptyset := 1$ and $\mathbf{u} \wedge 1 := \mathbf{u}$ for any multivector $\mathbf{u}$.

Theorem **5.1.** — *Suppose that* A_1 *and* A_2 *are (possibly empty) subsets of a finite set* E. *We have*

$$\mathbf{P}^H[A_1 \subseteq \mathfrak{B}, A_2 \cap \mathfrak{B} = \emptyset] = \big(P_{\mathrm{Ext}(H)}\theta_{A_1} \wedge P_{\mathrm{Ext}(H^\perp)}\theta_{A_2},\ \theta_{A_1} \wedge \theta_{A_2}\big)\,. \tag{5.2}$$

In particular, for any $A \subseteq E$, *we have*

$$\mathbf{P}^H[A \subseteq \mathfrak{B}] = \|P_{\mathrm{Ext}(H)}\theta_A\|^2\,. \tag{5.3}$$

Remark **5.2.** — The property (5.2) (in an equivalent form) appears in several places in the literature, including Theorem 3.1 of Shirai and Takahashi (2000) and Proposition A.8 of Borodin, Okounkov, and Olshanski (2000). Usually, it is derived from a different definition of the measure $\mathbf{P}^H$. For the uniform measure on bases of regular matroids, the case of (5.3) where $|A| = 2$ is Theorem 2.1 of Feder and Mihail (1992). The general case is related to Theorem 5.2.1 of Mehta (1991).

Proof. — Both sides of (5.2) are clearly 0 unless $A_1 \cap A_2 = \emptyset$, so we assume this disjointness from now on. Let $r := \dim H$.

Consider next the case where $A_1 \cup A_2 = E$. The left-hand side is 0 unless $|A_1| = r$ since every base has r elements. Also the right-hand side is 0 except in such a case since Ext(H) has multivectors only of rank at most r and $\mathrm{Ext}(H^\perp)$ has multivectors only of rank at most $|E| - r$. It remains to establish equality when $|A_1| = r$. In that case, we have

$$\begin{aligned} P_{\mathrm{Ext}(H)}\theta_{A_1} \wedge \theta_{A_2} &= P_{\mathrm{Ext}(H)}\theta_{A_1} \wedge \bigwedge_{e\in A_2} \big(P_H e + P_H^\perp e\big) \\ &= P_{\mathrm{Ext}(H)}\theta_{A_1} \wedge P_{\mathrm{Ext}(H^\perp)}\theta_{A_2}\,, \end{aligned}$$

as we see by expanding the product, using the fact that Ext(H) has multivectors only of rank at most r, and using Lemma 4.1. Therefore,

$$\begin{aligned} (P_{\mathrm{Ext}(H)}\theta_{A_1} \wedge P_{\mathrm{Ext}(H^\perp)}\theta_{A_2},\ \theta_{A_1} \wedge \theta_{A_2}) &= (P_{\mathrm{Ext}(H)}\theta_{A_1} \wedge \theta_{A_2},\ \theta_{A_1} \wedge \theta_{A_2}) \\ &= (P_{\mathrm{Ext}(H)}\theta_{A_1},\ \theta_{A_1}) \\ &= \mathbf{P}^H[\mathfrak{B} = A_1] \\ &= \mathbf{P}^H[A_1 \subseteq \mathfrak{B}, A_2 \cap \mathfrak{B} = \emptyset]\,. \end{aligned}$$

This establishes the result when E is partitioned as $A_1 \cup A_2$. In the general case, we can decompose over partitions to write

$$\begin{aligned}&\mathbf{P}^{H}[A_1 \subseteq \mathfrak{B}, A_2 \cap \mathfrak{B} = \emptyset] \\ &\quad = \sum_{C \subseteq E\setminus(A_1 \cup A_2)} \mathbf{P}^{H}\big[A_1 \cup C \subseteq \mathfrak{B}, \big(A_2 \cup (E \setminus C)\big) \cap \mathfrak{B} = \emptyset\big]\end{aligned} \tag{5.4}$$

and we can also write

$$\begin{aligned}\Big(P_{\mathrm{Ext}(H)}\theta_{A_1} \wedge P_{\mathrm{Ext}(H^{\perp})}\theta_{A_2},\ \theta_{A_1} \wedge \theta_{A_2}\Big) &= \Big(\bigwedge_{e\in A_1} P_H e \wedge \bigwedge_{e\in A_2} P_H^{\perp} e,\ \theta_{A_1} \wedge \theta_{A_2}\Big) \\ &= \Big(\bigwedge_{e\in A_1} P_H e \wedge \bigwedge_{e\in A_2} P_H^{\perp} e \wedge \bigwedge_{e\notin A_1\cup A_2} e,\ \theta_{A_1} \wedge \theta_{A_2} \wedge \bigwedge_{e\notin A_1\cup A_2} e\Big) \\ &= \Big(\bigwedge_{e\in A_1} P_H e \wedge \bigwedge_{e\in A_2} P_H^{\perp} e \wedge \bigwedge_{e\notin A_1\cup A_2} \big(P_H e + P_H^{\perp} e\big),\ \theta_{A_1} \wedge \theta_{A_2} \wedge \bigwedge_{e\notin A_1\cup A_2} e\Big) \\ &= \sum_{C\subseteq E\setminus(A_1\cup A_2)} \Big(\bigwedge_{e\in A_1\cup C} P_H e \wedge \bigwedge_{e\in A_2\cup(E\setminus C)} P_H^{\perp} e,\ \theta_{A_1\cup C} \wedge \theta_{A_2\cup(E\setminus C)}\Big).\end{aligned} \tag{5.5}$$

For each C, the summand on the right of (5.4) is equal to that on the right of (5.5) by what we have shown, whence the general case follows. □

We see immediately the relationship of orthogonality to duality:

Corollary **5.3.** — *If a subspace* $H \subseteq \ell^2(E)$ *represents a matroid* $\mathscr{M}$*, then its orthogonal complement* $H^{\perp}$ *represents the dual matroid* $\mathscr{M}^{\perp}$*:*

$$\forall B \in 2^{E} \qquad \mathbf{P}^{H^{\perp}}[E \setminus B] = \mathbf{P}^{H}[B]\,. \tag{5.6}$$

We now verify (2.1) and (2.2). Resuming the notation used there, we let the ith row of M be m_i. For some constant c, we thus have

$$\xi_H = c \bigwedge_{i=1}^{r} m_i\,, \tag{5.7}$$

whence by (4.1),

$$\mathbf{P}^{H}[B] = |(\xi_H,\ \theta_B)|^2 = |c|^2 \left|\det\,[(m_i,\ e)]_{i\le r,\ e\in B}\right|^2 = |c|^2 |\det M_B|^2\,.$$

To complete the derivation of (2.1), we must calculate $|c|^2$. For this, note that

$$1 = \|\xi_H\|^2 = |c|^2 \Big\| \bigwedge_{i=1}^{r} m_i \Big\|^2 = |c|^2 \det\big[(m_i,\ m_j)\big]_{i,j\le r} = |c|^2 \det(M_{(r)} M_{(r)}^{*})\,.$$

We record explicitly the following consequence noted in Sect. 1:

Corollary **5.4.** — *A matrix* M *is a coordinatization matrix of a matroid* $\mathscr{M} = (\mathrm{E}, \mathscr{B})$ *iff the row space* H *of* M *is a subspace representation of* $\mathscr{M}$.

This is also obvious from the following equivalences that hold for any scalars a_e:

$$\forall i \ \sum_e a_e(m_i, e) = 0 \iff \sum_e a_e e \in \ker \mathrm{M} = \mathrm{H}^\perp \iff \mathrm{P_H} \sum_e a_e e = 0$$
$$\iff \sum_e a_e \mathrm{P_H} e = 0 \,.$$

By Corollary 5.4, it follows that ★ is a subspace representation of the graphic matroid, as mentioned in Sect. 1.

In order to demonstrate (2.2), first note that row operations on M do not change the row space, H. Now let $\langle w_i ;\ 1 \le i \le r\rangle \in \ell^2(\mathrm{E})$ be the rows of $\widehat{\mathrm{M}}$ and let the entries be $\widehat{\mathrm{M}}_{i,e} = (w_i, e)$. Then

$$\mathrm{P_H} e = \sum_{i=1}^r (e, w_i) w_i = \sum_{i=1}^r \overline{\widehat{\mathrm{M}}_{i,e}} w_i \,,$$

whence $(\mathrm{P_H} e, e') = \sum_{i=1}^r \overline{\widehat{\mathrm{M}}_{i,e}} (w_i, e') = \sum_{i=1}^r \overline{\widehat{\mathrm{M}}_{i,e}} \widehat{\mathrm{M}}_{i,e'} = (\widehat{\mathrm{M}}^* \widehat{\mathrm{M}})_{e,e'}$. Since $(\mathrm{P_H} e, e')$ is the (e', e)-entry of the matrix of $\mathrm{P_H}$, we obtain (2.2).

We call a matroid **complex unimodular** if it has a coordinatization matrix M over **C** with $|\det \mathrm{M_B}| = 1$ for all bases B. These are known to be the same as the **sixth-root-of-unity** matroids, where the condition now is that it has a coordinatization matrix M over **C** with $(\det \mathrm{M_B})^6 = 1$ for all bases B (see Choe, Oxley, Sokal, and Wagner (2003) for this and other characterizations of such matroids).

Corollary **5.5.** — *A matroid* $\mathscr{M} = (\mathrm{E}, \mathscr{B})$ *has a representation via a subspace* $\mathrm{H} \subseteq \ell^2(\mathrm{E};\ \mathbf{R})$ *with* $\mathbf{P}^{\mathrm{H}}$ *equal to uniform measure on* $\mathscr{B}$ *iff* $\mathscr{M}$ *is a regular matroid. A matroid* $\mathscr{M} = (\mathrm{E}, \mathscr{B})$ *has a representation via a subspace* $\mathrm{H} \subseteq \ell^2(\mathrm{E};\ \mathbf{C})$ *with* $\mathbf{P}^{\mathrm{H}}$ *equal to uniform measure on* $\mathscr{B}$ *iff* $\mathscr{M}$ *is a complex unimodular matroid.*

Proof. — Having a subspace representation that gives uniform measure is equivalent to having a coordinatization matrix M with the above matrices $\mathrm{M_B}$ having determinants equal in absolute value for $\mathrm{B} \in \mathscr{B}$. In case the field is **R**, this is equivalent to requiring all $\det \mathrm{M_B} = \pm 1$. From the above, we see that this is the same as having some H representing $\mathscr{B}$ and some constant c with $c\xi_{\mathrm{H}}$ having all its coefficients in $\{0, \pm 1\}$. This condition is equivalent to regularity: see (3) and (4) of Theorem 3.1.1 of White (1987).

The statement for complex representations is clear from the definition. □

Remark **5.6.** — In dealing with electrical networks that have general conductances (not necessarily all 1), one uses not the uniform spanning tree measure, but the weighted spanning tree measure, where the probability of a tree is proportional to the product of the conductances of its edges. If the conductance, or weight, of the edge e is $w(e)$, then one defines the star at a vertex x by $\star_x := \sum_{e\in\mathsf{E}} \sqrt{w(e)}a(x, e)e$ and one defines $\bigstar$ and $\diamondsuit$ accordingly. The weighted spanning tree measure is then $\mathbf{P}^{\bigstar}$. In fact, given any positive weight function w on a ground set E and given any measure $\mathbf{P}^{\mathrm{H}}$ on a matroid, there is another subspace H_w such that for all bases B, we have $\mathbf{P}^{\mathrm{H}_w}[\mathrm{B}] = \mathrm{Z}^{-1}\mathbf{P}^{\mathrm{H}}[\mathrm{B}] \prod_{e\in\mathrm{B}} w(e)$, where Z is a normalizing constant. To see this, define the linear transformation D_w on $\ell^2(\mathrm{E})$ by $\mathrm{D}_w(e) = \sqrt{w(e)}e$. Then it is not hard to verify that the image H_w of H under D_w has this property. Similarly, if the matroid is represented by the matrix M with row space H, then the matrix MD_w represents the same matroid but with the transformed measure $\mathbf{P}^{\mathrm{H}_w}$.

6. Stochastic domination and conditioning

Let 2^{E} denote the collection of all subsets of E. Any probability measure $\mathbf{P}$ on $\mathscr{B}$ extends to a probability measure on the Borel σ-field of 2^{E} by setting $\mathbf{P}(\mathscr{A}) := \mathbf{P}(\mathscr{A} \cap \mathscr{B})$. In this section, we compare different probability measures on 2^{E} with E finite. First, for E finite, to each event $\mathscr{A} \subseteq 2^{\mathrm{E}}$, we associate the subspace

$$\mathrm{S}_{\mathscr{A}} := \text{the linear span of } \big\{\theta_{\mathrm{C}}\,;\ \mathrm{C} \in \mathscr{A}\big\} \subseteq \mathrm{Ext}(\mathrm{E})\,.$$

Note that C need not be in $\mathscr{B}$ here. An event $\mathscr{A}$ is called **increasing** if whenever $\mathrm{A} \in \mathscr{A}$ and $e \in \mathrm{E}$, we have also $\mathrm{A} \cup \{e\} \in \mathscr{A}$. Thus, $\mathscr{A}$ is increasing iff the subspace $\mathrm{S}_{\mathscr{A}}$ is an ideal (with respect to exterior product).

The probability measure $\mathbf{P}^{\mathrm{H}}$ clearly satisfies

$$\mathbf{P}^{\mathrm{H}}(\mathscr{A}) = \|\mathrm{P}_{\mathrm{S}_{\mathscr{A}}}\xi_{\mathrm{H}}\|^2$$

for any event $\mathscr{A}$.

Lemma **6.1.** — *Let* $\mathrm{S} \subseteq \mathrm{Ext}(\mathrm{E})$ *be an ideal,* $\mathbf{u} = \bigwedge_{i=1}^{k} u_i$, *and* $\mathbf{v} = \bigwedge_{j=1}^{l} v_j$. *If* $(u_i,\ v_j) = 0$ *for all* i *and* j, *then*

$$\big((\mathrm{P_S}\mathbf{u}) \wedge \mathbf{v},\ \mathrm{P_S}(\mathbf{u} \wedge \mathbf{v})\big) = \|\mathrm{P_S}\mathbf{u}\|^2\|\mathbf{v}\|^2\,.$$

Proof. — Since $(\mathrm{P_S}\mathbf{u}) \wedge \mathbf{v} \in \mathrm{S}$, we have

$$\begin{aligned}\big((\mathrm{P_S}\mathbf{u}) \wedge \mathbf{v},\ \mathrm{P_S}(\mathbf{u} \wedge \mathbf{v})\big) &= \big(\mathrm{P_S}\big((\mathrm{P_S}\mathbf{u}) \wedge \mathbf{v}\big),\ \mathbf{u} \wedge \mathbf{v}\big)\\ &= \big((\mathrm{P_S}\mathbf{u}) \wedge \mathbf{v},\ \mathbf{u} \wedge \mathbf{v}\big)\\ &= (\mathrm{P_S}\mathbf{u},\ \mathbf{u})(\mathbf{v},\ \mathbf{v})\\ &= \|\mathrm{P_S}\mathbf{u}\|^2\|\mathbf{v}\|^2\,.\end{aligned}$$

□

Given two probability measures $\mathbf{P}^1$, $\mathbf{P}^2$ on 2^{E}, we say that $\mathbf{P}^2$ **stochastically dominates** $\mathbf{P}^1$ and write $\mathbf{P}^1 \preccurlyeq \mathbf{P}^2$ if for all increasing events $\mathscr{A}$, we have $\mathbf{P}^1(\mathscr{A}) \le \mathbf{P}^2(\mathscr{A})$.

Theorem **6.2.** — *Let* E *be finite and let* $\mathrm{H}_1 \subset \mathrm{H}_2$ *be subspaces of* $\ell^2(\mathrm{E})$. *Then* $\mathbf{P}^{\mathrm{H}_1} \preccurlyeq \mathbf{P}^{\mathrm{H}_2}$.

Proof. — Let $\mathscr{A}$ be an increasing event. Take an orthonormal basis $\langle u_i\,;\ i \le r_2\rangle$ of H_2 such that $\langle u_i\,;\ i \le r_1\rangle$ is an orthonormal basis of H_1 and such that $\xi_{\mathrm{H}_j} = \bigwedge_{i=1}^{r_j} u_i$ for $j = 1, 2$. Apply Lemma 6.1 to $\mathbf{u} := \bigwedge_{i=1}^{r_1} u_i$ and $\mathbf{v} := \bigwedge_{i=r_1+1}^{r_2} u_i$ to conclude that

$$\begin{aligned}\|\mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_1}\|^2 &= \|\mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_1}\|^2\|\mathbf{v}\|^2 = \big((\mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_1}) \wedge \mathbf{v},\ \mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_2}\big)\\ &\le \|\mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_1} \wedge \mathbf{v}\|\|\mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_2}\| \le \|\mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_1}\|\|\mathbf{v}\|\|\mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_2}\|\\ &= \|\mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_1}\|\|\mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_2}\|\end{aligned}$$

by (4.2), whence $\|\mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_1}\| \le \|\mathrm{P}_{\mathrm{S}_\mathscr{A}}\xi_{\mathrm{H}_2}\|$, i.e., $\mathbf{P}^{\mathrm{H}_1}(\mathscr{A}) \le \mathbf{P}^{\mathrm{H}_2}(\mathscr{A})$. □

A **minor** of a matroid $\mathscr{M} = (\mathrm{E}, \mathscr{B})$ is one obtained from $\mathscr{M}$ by repeated contraction and deletion: Given $e \in \mathrm{E}$, the **contraction** of $\mathscr{M}$ along e is $\mathscr{M}/e := (\mathrm{E}, \mathscr{B}/e)$, where $\mathscr{B}/e := \{\mathrm{B} \in \mathscr{B}\,;\ e \in \mathrm{B}\}$, while the **deletion** of $\mathscr{M}$ along e is $\mathscr{M} \setminus e := (\mathrm{E}, \mathscr{B} \setminus e)$, where $\mathscr{B} \setminus e := \{\mathrm{B} \in \mathscr{B}\,;\ e \notin \mathrm{B}\}$. Note that, contrary to usual convention, we are keeping the same ground set, E.

For any $\mathrm{F} \subseteq \mathrm{E}$, let $[\mathrm{F}]$ be the closure of the linear span of the unit vectors $\{e\,;\ e \in \mathrm{F}\}$. We shall also write $[u]$ for the subspace spanned by any vector u.

Proposition **6.3.** — *Let* E *be finite and* H *be a subspace of* $\ell^2(\mathrm{E})$. *For any* $e \in \mathrm{E}$ *with* $0 < \mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B}] < 1$, *we have*

$$\mathbf{P}^{\mathrm{H}}(\,\bullet \mid e \in \mathfrak{B}) = \mathbf{P}^{(\mathrm{H}\cap e^\perp)+[e]}(\,\bullet\,)$$

and

$$\mathbf{P}^{\mathrm{H}}(\,\bullet \mid e \notin \mathfrak{B}) = \mathbf{P}^{(\mathrm{H}+[e])\cap e^\perp}(\,\bullet\,)\,.$$

In particular, $(\mathrm{H} \cap e^\perp) + [e]$ *represents* $\mathscr{M}/e$ *and* $(\mathrm{H} + [e]) \cap e^\perp$ *represents* $\mathscr{M} \setminus e$. *Moreover, signs (or, in the complex case, unit scalar factors) can be chosen for* $\xi_{(\mathrm{H}\cap e^\perp)+[e]}$ *and* $\xi_{(\mathrm{H}+[e])\cap e^\perp}$ *so that for all* $\mathrm{B} \in \mathscr{B}$,

$$(\mathbf{6.1})\qquad \big(\theta_{\mathrm{B}},\ \xi_{(\mathrm{H}\cap e^\perp)+[e]}\big) = \begin{cases}(\theta_{\mathrm{B}},\ \xi_{\mathrm{H}})/\|\mathrm{P}_{\mathrm{H}}e\| & \textit{if } e \in \mathrm{B},\\ 0 & \textit{if } e \notin \mathrm{B}\end{cases}$$

and

$$(\mathbf{6.2})\qquad \big(\theta_{\mathrm{B}},\ \xi_{(\mathrm{H}+[e])\cap e^\perp}\big) = \begin{cases}(\theta_{\mathrm{B}},\ \xi_{\mathrm{H}})/\big\|\mathrm{P}_{\mathrm{H}}^\perp e\big\| & \textit{if } e \notin \mathrm{B},\\ 0 & \textit{if } e \in \mathrm{B}.\end{cases}$$

Proof. — Clearly $(\theta_B,\ \xi_H) = \big(\theta_B,\ (\xi_H \vee e) \wedge e\big)$ when $e \in B$, while the right-hand side of this equation is 0 for $e \notin B$. Applying (4.4) followed by (4.3), we obtain (6.1). The proof of (6.2) is similar. These imply the formulas for the conditional probabilities. □

Given a disjoint pair of possibly infinite sets $A, B \subseteq E$, define

$$H_{A,B} := \big((H \cap A^\perp) + [A \cup B]\big) \cap B^\perp . \tag{6.3}$$

It is straightforward to verify that

$$H_{A,B} = \big((H + [B]) \cap (A \cup B)^\perp\big) + [A] . \tag{6.4}$$

Indeed, denote by $H'_{A,B}$ the right-hand side of (6.4). Suppose that $u \in H_{A,B}$ and write $u = u_1 + u_2 + u_3$ with $u_1 \in H \cap A^\perp$, $u_2 \in [A]$, and $u_3 \in [B]$. Since $u_1 \in H$, it follows that $u_1 + u_3 \in H + [B]$. Since $u_1 \in A^\perp$ and $u_3 \in [B] \subseteq A^\perp$, we obtain $u_1 + u_3 \in (H + [B]) \cap A^\perp$. Since $u_2 \in [A] \subseteq B^\perp$ and $u \in B^\perp$, we have that $u_1 + u_3 = u - u_2 \in B^\perp$, whence $u_1 + u_3 \in \big((H + [B]) \cap A^\perp\big) \cap B^\perp = (H + [B]) \cap (A \cup B)^\perp$. Since $u_2 \in [A]$, it follows that $u \in H'_{A,B}$.

Conversely, suppose that $u \in H'_{A,B}$ and write $u = u_1 + u_2 + u_3$ with $u_1 \in H$, $u_2 \in [B]$, $u_3 \in [A]$, and $u_1 + u_2 \in [A \cup B]^\perp$. Since $u_1 + u_2 \in A^\perp$ and $u_2 \in A^\perp$, it follows that $u_1 \in A^\perp$, whence $u_1 \in H \cap A^\perp$ and so $u \in (H \cap A^\perp) + [A \cup B]$. Since $u_1 + u_2 \in B^\perp$ and $u_3 \in B^\perp$, we also have $u \in B^\perp$, whence $u \in H_{A,B}$, as desired.

Corollary **6.4.** — *For any closed subspace* H *and* $A, B, C, D \subseteq E$ *with* $(A \cup B) \cap (C \cup D) = \emptyset$, *we have*

$$(H_{A,B})_{C,D} = H_{A \cup C, B \cup D} .$$

Proof. — In showing (6.4), we have shown that $H_{A,B} = (H_{A,\emptyset})_{\emptyset,B} = (H_{\emptyset,B})_{A,\emptyset}$. Also, we have

$$\begin{aligned}
(H_{A,\emptyset})_{B,\emptyset} &= \Big(\big((H \cap A^\perp) + [A]\big) \cap B^\perp\Big) + [B] \\
&= \Big(\big((H \cap A^\perp) + [A \setminus B]\big) \cap (B \setminus A)^\perp\Big) + [B] \\
&= \big((H \cap (A \cup B)^\perp) + [A \setminus B]\big) + [B] \\
&= (H \cap (A \cup B)^\perp) + [A \cup B] = H_{A \cup B, \emptyset} ,
\end{aligned}$$

whence also $(H_{\emptyset,A})_{\emptyset,B} = H_{\emptyset,A \cup B}$ by duality. Therefore,

$$\begin{aligned}
(H_{A,B})_{C,D} &= (((H_{A,\emptyset})_{\emptyset,B})_{\emptyset,D})_{C,\emptyset} = ((H_{A,\emptyset})_{\emptyset,B \cup D})_{C,\emptyset} \\
&= ((H_{A,\emptyset})_{C,\emptyset})_{\emptyset,B \cup D} = (H_{A \cup C})_{\emptyset,B \cup D} = H_{A \cup C, B \cup D} .
\end{aligned}$$

□

We remark that when $A \cup B$ is finite, the subspace $H_{A,B}$ is closed, since the sum of two closed subspaces, one of finite dimension, is always closed (Halmos (1982), Problem 13) and since the intersection of closed subspaces is always closed.

It follows by induction from Proposition 6.3 that for *finite* E, if $\mathbf{P}^H[A \subseteq \mathfrak{B}, B \cap \mathfrak{B} = \emptyset] > 0$, then

$$\mathbf{P}^H(\,\bullet \mid A \subseteq \mathfrak{B}, B \cap \mathfrak{B} = \emptyset) = \mathbf{P}^{H_{A,B}}(\,\bullet\,)\,. \tag{6.5}$$

For a set $K \subseteq E$, let $\mathscr{F}(K)$ denote the σ-field generated by the events $\{e \in \mathfrak{B}\}$ for $e \in K$. We shall say that the events in $\mathscr{F}(K)$ are **measurable with respect to** K and likewise for functions that are measurable with respect to $\mathscr{F}(K)$. We also say that an event or a function that is measurable with respect to $E \setminus \{e\}$ **ignores** e. Thus, an event $\mathscr{A}$ ignores e iff for all $\mathbf{u} \in S_{\mathscr{A}}$, also $\mathbf{u} \wedge e \in S_{\mathscr{A}}$ and $\mathbf{u} \vee e \in S_{\mathscr{A}}$. We say that a probability measure $\mathbf{P}$ on 2^E has **negative associations** if

$$\mathbf{P}(\mathscr{A}_1 \cap \mathscr{A}_2) \leq \mathbf{P}(\mathscr{A}_1)\mathbf{P}(\mathscr{A}_2)$$

whenever $\mathscr{A}_1$ and $\mathscr{A}_2$ are increasing events that are measurable with respect to complementary subsets of E. A function $f : 2^E \to \mathbf{R}$ is called **increasing** if for all $A \in 2^E$ and all $e \in E$, we have $f(A \cup \{e\}) \geq f(A)$. It is well known and not hard to see that $\mathbf{P}$ has negative associations iff for any pair f_1, f_2 of increasing functions that are measurable with respect to complementary subsets of E,

$$\mathbf{E}[f_1 f_2] \leq \mathbf{E}[f_1]\mathbf{E}[f_2]\,. \tag{6.6}$$

In this case, for any collection $f_1, f_2, \ldots, f_n$ of increasing *nonnegative* functions that are measurable with respect to pairwise disjoint subsets of E, we have

$$\mathbf{E}[f_1 f_2 \cdots f_n] \leq \mathbf{E}[f_1]\mathbf{E}[f_2] \cdots \mathbf{E}[f_n]\,. \tag{6.7}$$

This is shown by an easy induction argument. One could replace "increasing" by "decreasing" just as well.

The following result was proved by Feder and Mihail (1992), Lemma 3.2, for the uniform measure on bases of "balanced" matroids, a class of matroids including regular ones.

Theorem **6.5.** — *Let* E *be finite. If* $\mathscr{A}$ *is an increasing event that ignores* e *and* $\mathbf{P}^H[e \in \mathfrak{B}] > 0$, *then* $\mathbf{P}^H(\mathscr{A} \mid e \in \mathfrak{B}) \leq \mathbf{P}^H(\mathscr{A})$. *More generally,* $\mathbf{P}^H$ *has negative associations.*

Proof. — We have $\mathbf{P}^H(\mathscr{A} \mid e \in \mathfrak{B}) = \mathbf{P}^{(H \cap e^\perp)+[e]}(\mathscr{A}) = \mathbf{P}^{(H \cap e^\perp)}(\mathscr{A})$ since the effect of having e in a subspace is simply to make $e \in \mathfrak{B}$ with probability 1 and since $\mathscr{A}$ ignores e. Since $H \cap e^\perp \subseteq H$, it follows from Theorem 6.2 that $\mathbf{P}^{(H \cap e^\perp)}(\mathscr{A}) \leq \mathbf{P}^H(\mathscr{A})$, which proves the first assertion.

For the more general statement, we follow the method of proof of Feder and Mihail (1992). This uses induction on the cardinality of E. The case $|\mathrm{E}| = 1$ is trivial. Let $r := \dim \mathrm{H}$. Given $\mathscr{A}_1$ and $\mathscr{A}_2$ as specified with $\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1), \mathbf{P}^{\mathrm{H}}(\mathscr{A}_2) > 0$, we have

$$\sum_{e\in \mathrm{E}} \mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B}] = \mathbf{E}^{\mathrm{H}}\big[|\mathfrak{B}|\big] = r = \mathbf{E}^{\mathrm{H}}\big[|\mathfrak{B}| \bigm| \mathscr{A}_1\big] = \sum_{e\in \mathrm{E}} \mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B} \mid \mathscr{A}_1]$$

since $|\mathfrak{B}| = r$ $\mathbf{P}^{\mathrm{H}}$-a.s. By the preceding paragraph, $\mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B} \mid \mathscr{A}_1] \le \mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B}]$ for those e ignored by $\mathscr{A}_1$. It follows that the opposite inequality holds for some e not ignored by $\mathscr{A}_1$ and with $\mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B}] > 0$. Fix such an $e \in \mathrm{E}$. Note that $\mathscr{A}_2$ ignores e. We have $\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid e \in \mathfrak{B}) \ge \mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid e \notin \mathfrak{B})$, where the right-hand side is defined to be 0 if $\mathbf{P}[e \notin \mathfrak{B}] = 0$. Now

$$\begin{aligned} \mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid \mathscr{A}_2) = {} & \mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B} \mid \mathscr{A}_2]\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid \mathscr{A}_2, e \in \mathfrak{B}) \\ & + \mathbf{P}^{\mathrm{H}}[e \notin \mathfrak{B} \mid \mathscr{A}_2]\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid \mathscr{A}_2, e \notin \mathfrak{B}) \,. \end{aligned} \tag{6.8}$$

The induction hypothesis and Proposition 6.3 imply that (6.8) is at most

$$\mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B} \mid \mathscr{A}_2]\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid e \in \mathfrak{B}) + \mathbf{P}^{\mathrm{H}}[e \notin \mathfrak{B} \mid \mathscr{A}_2]\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid e \notin \mathfrak{B})\,; \tag{6.9}$$

this is because by Proposition 6.3, the two measures conditioned on whether or not e lies in $\mathfrak{B}$ can each be regarded as measures arising from orthogonal projections on $\ell^2(\mathrm{E} \setminus \{e\})$ and because $\mathscr{A}_1$ and $\mathscr{A}_2$ each transform to increasing events in $2^{\mathrm{E}\setminus\{e\}}$. By what we have proved in the first paragraph, we have that $\mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B} \mid \mathscr{A}_2] \le \mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B}]$ and we have chosen e so that $\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid e \in \mathfrak{B}) \ge \mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid e \notin \mathfrak{B})$. Therefore, (6.9) is at most

$$\mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B}]\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid e \in \mathfrak{B}) + \mathbf{P}^{\mathrm{H}}[e \notin \mathfrak{B}]\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid e \notin \mathfrak{B}) = \mathbf{P}^{\mathrm{H}}(\mathscr{A}_1) \tag{6.10}$$

since the quantity in (6.10) minus that in (6.9) equals

$$\big(\mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B}] - \mathbf{P}^{\mathrm{H}}[e \in \mathfrak{B} \mid \mathscr{A}_2]\big)\big(\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid e \in \mathfrak{B}) - \mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid e \notin \mathfrak{B})\big) \ge 0\,.$$

This completes the induction step and hence the proof. ◻

It seems that there should be an elegant proof of Theorem 6.5 along the lines of the proof of Theorem 6.2, but we were not able to find one. However, the special case of Theorem 6.5 saying that

$$\mathbf{P}^{\mathrm{H}}[\mathrm{A} \cup \mathrm{B} \subset \mathfrak{B}] \le \mathbf{P}^{\mathrm{H}}[\mathrm{A} \subset \mathfrak{B}]\mathbf{P}^{\mathrm{H}}[\mathrm{B} \subset \mathfrak{B}]$$

is an immediate consequence of (4.2) for $\mathbf{u} := \bigwedge_{e\in \mathrm{A}} P_{\mathrm{H}} e$ and $\mathbf{v} := \bigwedge_{e\in \mathrm{B}} P_{\mathrm{H}} e$. This special case was also proved in Proposition 2.7 of Shirai and Takahashi (2003).

Combining Theorem 6.5 with Equation (6.5), we obtain the following result:

Corollary **6.6.** — *If* E *is finite,* A *and* B *are disjoint subsets of* E, *and* H *is a subspace of* $\ell^2(\mathrm{E})$, *then the conditional probability measure* $\mathbf{P}^{\mathrm{H}}(\,\bullet \mid \mathrm{A} \subseteq \mathfrak{B}, \mathrm{B} \cap \mathfrak{B} = \emptyset)$ *has negative associations.*

In the language of Pemantle (2000), this corollary says that the measure $\mathbf{P}^{\mathrm{H}}$ is conditionally negatively associated (CNA). In fact, it enjoys the strongest property studied by Pemantle (2000), namely, CNA with so-called external fields. This follows by Remark 5.6.

Soshnikov (2000b) proves a very general central limit theorem for determinantal probability measures (and determinantal point processes). Many theorems for independent random variables are known to hold for negatively associated random variables, especially for a stationary collection of negatively random variables indexed by $\mathbf{Z}^d$. For example, see Newman (1984), Shao and Su (1999), Shao (2000), Zhang and Wen (2001), Zhang (2001), and the references therein. We shall merely state one of the easiest of these theorems, as it will prove useful to us later:

Corollary **6.7.** — *Let* E *be finite,* H *be a subspace of* $\ell^2(\mathrm{E})$*, and* $\mathrm{A} \subseteq \mathrm{E}$. *Write* $\mu := \mathbf{E}^{\mathrm{H}}[|\mathfrak{B} \cap \mathrm{A}|]$. *Then for any* $a > 0$*, we have*

$$\mathbf{P}^{\mathrm{H}}\Big[\big||\mathfrak{B} \cap \mathrm{A}| - \mu\big| \geq a\Big] \leq 2e^{-2a^2/|\mathrm{A}|}\,. \tag{6.11}$$

Essentially the same observation is made by Dubhashi and Ranjan (1998). The standard proof, with the addition of (6.7) to replace independence, applies. See, e.g., Alon and Spencer (2001), Corollary A.1.7, for such a proof for independent random variables.

Recall Kirchhoff's theorem that the current $\mathrm{Y}(e, e)$ is the probability that e lies in a uniform spanning tree of a graph G. Kirchhoff extended this equation to express the entire current vector in a network by a sum over spanning trees; see Thomassen (1990) for a statement and short combinatorial proof of this result. Such a combinatorial interpretation and proof extends to the uniform measure on bases of regular matroids, but not, as far as we can tell, to our general case. We do, however, have the following extension, in which (6.13) is related to, but seems not to follow from, Maurer (1976), Theorem 1.

Proposition **6.8.** — *Let* E *be finite. For any* $v \in \ell^2(\mathrm{E})$ *and* $\mathrm{B} \in \mathscr{B}$*, define* $a_v(e, \mathrm{B}) = a_v(e, \mathrm{B}; \mathrm{H})$ *by*

$$\mathrm{P_H}v = \sum_{e\in \mathrm{B}} a_v(e, \mathrm{B})\mathrm{P_H}e\,; \tag{6.12}$$

this uniquely determines the coefficients $a_v(e, \mathrm{B})$ *since the vectors* $\langle \mathrm{P_H}e\,;\ e \in \mathrm{B}\rangle$ *form a basis of* H. *Put*

$$\zeta^v_{\mathrm{B}} := \zeta^v_{\mathrm{B}}(\mathrm{H}) := \sum_{e\in \mathrm{B}} a_v(e, \mathrm{B})e\,.$$

Then

$$(\mathbf{6.13}) \qquad P_H v = \mathbf{E}^H \zeta^v_{\mathfrak{B}} \,.$$

In addition, if $S \subseteq F \subset E$ *and* $e \notin F$, *then*

$$(\mathbf{6.14}) \qquad P_{H^F_S} e = P^{\perp}_{[F]} \mathbf{E}^H [\zeta^e_{\mathfrak{B}}(H) \mid \mathfrak{B} \cap F = S] \,,$$

where

$$(\mathbf{6.15}) \qquad H^F_S := \big(H + [F \setminus S]\big) \cap [F]^{\perp} \,.$$

In order to understand this proposition better, we interpret it for the graphical matroid of a finite graph G and $v = e \in \mathsf{E}$. In this case, we have $a_e(f, B) \in \{0, 1\}$ for every edge f and ζ^e_B is the path in B from the tail of e to the head of e. Equation (6.13) is then the theorem of Kirchhoff (1847) alluded to above. Equation (6.14) is the same thing applied to a minor of G.

Proof. — For $B \in \mathscr{B}$, let $\theta_B = \bigwedge_{i=1}^r e_i$. For any j, if we take the wedge product of (6.12) with $\bigwedge_{i<j} P_H e_i$ on the left and $\bigwedge_{i>j} P_H e_i$ on the right and use Lemma 4.1, we obtain

$$P_{\mathrm{Ext}(H)} \left(\bigwedge_{i<j} e_i \wedge v \wedge \bigwedge_{i>j} e_i \right) = a_v(e_j, B) P_{\mathrm{Ext}(H)} \theta_B \,,$$

which is the same as

$$P_{\mathrm{Ext}(H)}[(\theta_B \vee e_j) \wedge v] = a_v(e_j, B) P_{\mathrm{Ext}(H)} \theta_B \,.$$

Therefore,

$$(\mathbf{6.16}) \qquad a_v(e_j, B) = \big((\theta_B \vee e_j) \wedge v,\ \xi_H\big) / (\theta_B,\ \xi_H) \,.$$

Since $P^H[B] = (\xi_H,\ \theta_B)(\theta_B,\ \xi_H)$ by (5.1) and $\theta_B \vee e = 0$ for $e \notin B$, it follows that

$$\begin{aligned}
\mathbf{E}^H \zeta^v_{\mathfrak{B}} &= \sum_B (\xi_H,\ \theta_B) \sum_{e \in B} \big((\theta_B \vee e) \wedge v,\ \xi_H\big) e \\
&= \sum_B (\xi_H,\ \theta_B) \sum_{e \in E} \big(\theta_B,\ (\xi_H \vee v) \wedge e\big) e \\
&= \sum_{e \in E} e \sum_B (\xi_H,\ \theta_B) \big(\theta_B,\ (\xi_H \vee v) \wedge e\big) = \sum_{e \in E} e \big(\xi_H,\ (\xi_H \vee v) \wedge e\big) \\
&\qquad\qquad [\text{since } \theta_B \text{ are orthonormal and } \xi_H \text{ lies in their span}] \\
&= \sum_{e \in E} \big(\xi_H \vee e,\ \xi_H \vee v\big) e = \sum_{e \in E} (P_H v,\ e) e = P_H v
\end{aligned}$$

by (4.5). This proves (6.13).

To show (6.14), note that

$$\text{(6.17)} \qquad H_{S,F\setminus S} = H_S^F + [S]\,.$$

By (6.5), we have that $\mathbf{P}^H(\,\bullet \mid \mathfrak{B} \cap F = S) = \mathbf{P}^{H_{S,F\setminus S}}(\,\bullet\,)$. In fact, induction from Proposition 6.3 shows that $(\theta_B, \xi_{H_{S,F\setminus S}})/(\theta_B, \xi_H)$ does not depend on B so long as $B \cap F = S$; this quotient is 0 if $B \cap F \neq S$. Now $B \cap F = S$ and $e, e' \notin F$ imply that $\big((B \setminus e') \cup e\big) \cap F = S$. Inspection of (6.16) shows, therefore, that if $B \cap F = S$ and $e \notin F$, then $a_e(e', B; H) = a_e(e', B; H_{S,F\setminus S})$ for $e' \in B \setminus F$, and thus $P_{[F]}^{\perp}\zeta_B^e(H) = \sum_{e' \in B\setminus F} a_e(e', B; H)e' = \zeta_B^e(H_{S,F\setminus S})$. Therefore,

$$\begin{aligned} P_{[F]}^{\perp}\mathbf{E}^H[\zeta_B^e(H) \mid B \cap F = S] &= P_{[F]}^{\perp}\mathbf{E}^H[\zeta_B^e(H_{S,F\setminus S}) \mid B \cap F = S] \\ &= P_{[F]}^{\perp}\mathbf{E}^{H_{S,F\setminus S}}[\zeta_B^e(H_{S,F\setminus S})] = P_{[F]}^{\perp}P_{H_{S,F\setminus S}}e \\ &= P_{H_S^F}e\,. \end{aligned}$$

□

7. The infinite case

Most interesting results about uniform spanning trees and forests arise in the setting of infinite graphs. In this section and the following one, we shall see what can be accomplished for general determinantal probability measures on infinite ground sets E. We still restrict ourselves in this section to orthogonal projections. In the next section, we extend both the finite and infinite cases to positive contractions.

Let $|E| = \infty$ and consider first a finite-dimensional subspace H of $\ell^2(E)$. Let us order E as $\{e_i\,;\ i \geq 1\}$. Define H_k as the image of the orthogonal projection of H onto the span of $\{e_i\,;\ 1 \leq i \leq k\}$. By considering a basis of H, we see that $P_{H_k} \to P_H$ in the strong operator topology (SOT), i.e., for all $v \in \ell^2(E)$, we have $\|P_{H_k}v - P_Hv\| \to 0$ as $k \to \infty$. We shall write $H_k \xrightarrow{\text{SOT}} H$ for $P_{H_k} \to P_H$ in the SOT. It is also easy to see that if $r := \dim H$, then $\dim H_k = r$ for all large k and, in fact, $\xi_{H_k} \to \xi_H$ in the usual norm topology. It follows that (5.2) holds for this subspace H and for any finite $A_1, A_2 \subset E$.

Now let H be an infinite-dimensional (closed) subspace of $\ell^2(E)$. It is well known that if H_n are (closed) subspaces of $\ell^2(E)$ with $H_n \uparrow H$ (meaning that $H_n \subseteq H_{n+1}$ and $\bigcup H_n$ is dense in H) or with $H_n \downarrow H$ (meaning that $H_n \supseteq H_{n+1}$ and $\bigcap H_n = H$), then $H_n \xrightarrow{\text{SOT}} H$. [The proof follows immediately from writing H as the orthogonal direct sum of its subspaces $H_{n+1} \cap H_n^{\perp}$ in the first case and then by duality in the second.] Choose an increasing sequence of finite-dimensional subspaces $H_k \uparrow H$. Since $H_k \xrightarrow{\text{SOT}} H$, we have

$$\text{(7.1)} \qquad \text{for all finite sets A} \quad \det(P_{H_k}\restriction A) \to \det(P_H\restriction A)\,,$$

whence $\mathbf{P}^{H_k}$ has a weak* limit that we denote $\mathbf{P}^{H}$ and that satisfies (5.2). From this construction, we see that the statement of Theorem 6.2 is valid for two possibly infinite-dimensional subspaces, one contained in the other:

Theorem **7.1.** — *Let* E *be finite or infinite and let* $H_1 \subset H_2$ *be closed subspaces of* $\ell^2(E)$. *Then* $\mathbf{P}^{H_1} \preccurlyeq \mathbf{P}^{H_2}$.

We also note that for *any* sequence of subspaces H_k, if $H_k \xrightarrow{\text{SOT}} H$, then $\mathbf{P}^{H_k} \to \mathbf{P}^{H}$ weak* because (7.1) then holds.

Establishing a conjecture of BLPS (2001), Morris (2003) proved that on any network (G, w) (where G is the underlying graph and w is the function assigning conductances, or weights, to the edges), for $\mathsf{WSF}(G, w)$-a.e. forest $\mathfrak{F}$ and for every component tree T of $\mathfrak{F}$, the WSF of $(T, w\restriction T)$ equals T a.s. This suggests the following extension. Given a subspace H of $\ell^2(E)$ and a set $B \subseteq E$, the subspace of [B] "most like" or "closest to" H is the closure of the image of H under the orthogonal projection $P_{[B]}$; we denote this subspace by H_B. For example, if $H = \bigstar(G)$, then $H_B = \bigstar(B)$ since for each $x \in V(G)$, we have $P_{[B]}(\star_x^G) = \star_x^B$. To say that $\mathbf{P}^{H_B}$ is concentrated on {B} is the same as to say that $H_B = [B]$. This motivates the following theorem and shows how it is an extension of Morris's theorem.

Theorem **7.2.** — *For any closed subspace* H *of* $\ell^2(E)$, *we have* $H_{\mathfrak{B}} = [\mathfrak{B}]$ $\mathbf{P}^{H}$*-a.s.*

After the proof of this theorem, we shall give some applications. In order to establish Theorem 7.2, we shall use several short lemmas. The overall proof is modelled on the original (second) proof by Morris.[⋆]

Lemma **7.3.** — *For any closed subspace* H *and any* $B \subseteq E$, *we have* $H_B = \overline{(H + B^\perp) \cap [B]} = \overline{H + B^\perp} \cap [B]$.

Proof. — Define $H'_B := (H + B^\perp) \cap [B]$ and $H''_B := \overline{H + B^\perp} \cap [B]$. We shall show that $H_B \subseteq \overline{H'_B} \subseteq H''_B \subseteq H_B$. Given $u \in H$, write $u = u_1 + u_2$ with $u_1 \in [B]$ and $u_2 \in B^\perp$. Then $u_1 = u - u_2 \in H + B^\perp$, and so $P_{[B]}u = u_1 \in H'_B$. Therefore, $H_B \subseteq \overline{H'_B}$. It is clear that $H'_B \subseteq H''_B$ and that H''_B is closed, whence $\overline{H'_B} \subseteq H''_B$. Finally, given $u \in H''_B$, write $u = \lim_{n\to\infty} u^{(n)}$, with $u^{(n)} = u_1^{(n)} + u_2^{(n)}$, where $u_1^{(n)} \in H$ and $u_2^{(n)} \in B^\perp$. Since $u \in [B]$, we have

$$u = P_{[B]}u = P_{[B]} \lim_{n\to\infty} u^{(n)} = \lim_{n\to\infty} P_{[B]}u^{(n)} = \lim_{n\to\infty} P_{[B]}u_1^{(n)} \in H_B\,.$$

□

Lemma **7.4.** — *For any closed subspace* H *and any* $B \subseteq E$, *we have* $\overline{H_B + B^\perp} = \overline{H + B^\perp}$.

[⋆] His proof is much shorter than our proof as he could rely on known facts about electrical networks. In particular, the relationship of our proof to that of Morris may not be so apparent. The part of our proof that is easiest to recognize is Lemma 7.5, which is a version of Rayleigh's monotonicity principle for electric networks.

Proof. — For any $u \in \mathrm{H}$, write $u = u_1 + u_2$ with $u_1 \in [\mathrm{B}]$ and $u_2 \in \mathrm{B}^\perp$. We have that $u + \mathrm{B}^\perp = u_1 + \mathrm{B}^\perp = \mathrm{P}_{[\mathrm{B}]}u + \mathrm{B}^\perp$. Since the closure of $\{u + \mathrm{B}^\perp;\ u \in \mathrm{H}\}$ equals $\overline{\mathrm{H} + \mathrm{B}^\perp}$ and the closure of $\{\mathrm{P}_{[\mathrm{B}]}u + \mathrm{B}^\perp;\ u \in \mathrm{H}\}$ equals $\mathrm{H}_\mathrm{B} + \mathrm{B}^\perp$, the result follows. □

Lemma **7.5.** — *For any closed subspace* H *and* $\mathrm{B} \subseteq \mathrm{E}$, *if* $v \in [\mathrm{B}]$, *then* $\|\mathrm{P}_{\mathrm{H}_\mathrm{B}} v\| \ge \|\mathrm{P}_\mathrm{H} v\|$.

Proof. — We have $\mathrm{P}_{\mathrm{H}_\mathrm{B}} v = \mathrm{P}_{\mathrm{H}_\mathrm{B} + \mathrm{B}^\perp} v$ since $v \in [\mathrm{B}]$. By Lemma 7.4, it follows that $\mathrm{P}_{\mathrm{H}_\mathrm{B}} v = \mathrm{P}_{\overline{\mathrm{H}+\mathrm{B}^\perp}} v$. Since $\mathrm{H} \subseteq \overline{\mathrm{H} + \mathrm{B}^\perp}$, it follows that $\mathrm{P}_\mathrm{H} v = \mathrm{P}_\mathrm{H}\mathrm{P}_{\mathrm{H}_\mathrm{B}} v$, whence $\|\mathrm{P}_\mathrm{H} v\| \le \|\mathrm{P}_{\mathrm{H}_\mathrm{B}} v\|$, as desired. □

Remark **7.6.** — More generally, if $v \in [\mathrm{B}]$ and $\mathrm{B} \subseteq \mathrm{C} \subseteq \mathrm{E}$, then $\|\mathrm{P}_{\mathrm{H}_\mathrm{B}} v\| \ge \|\mathrm{P}_{\mathrm{H}_\mathrm{C}} v\|$. Indeed, since $\mathrm{P}_{[\mathrm{B}]}\mathrm{H} = \mathrm{P}_{[\mathrm{B}]}\mathrm{P}_{[\mathrm{C}]}\mathrm{H}$, we have $\mathrm{H}_\mathrm{B} = (\mathrm{H}_\mathrm{C})_\mathrm{B}$. Thus, the inequality follows from applying Lemma 7.5 to the space H_C, rather than to $\ell^2(\mathrm{E})$.

Lemma **7.7.** — *Fix* $e \in \mathrm{E}$. *Let* $\mathrm{B} \subseteq \mathrm{E}$ *and* H *be a closed subspace of* [B]. *Suppose that* $\mathrm{F}_n \subseteq \mathrm{E}$ *form an increasing sequence of sets with union* $\mathrm{E} \setminus \{e\}$. *If* $e \notin \mathrm{H}$, *then* $\lim_{n\to\infty} \|\mathrm{P}_{\mathrm{H}_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}}} e\| = 0$.

Proof. — Since $\mathrm{H} \subseteq [\mathrm{B}]$, we have by (6.3) that

$$\mathrm{H}_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}} = \big(\mathrm{H} \cap (\mathrm{F}_n \cap \mathrm{B})^\perp\big) + [\mathrm{F}_n \cap \mathrm{B}]\,,$$

whence

$$\mathrm{P}_{\mathrm{H}_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}}} e = \mathrm{P}_{\mathrm{H}\cap(\mathrm{F}_n\cap\mathrm{B})^\perp} e + \mathrm{P}_{[\mathrm{F}_n\cap\mathrm{B}]} e = \mathrm{P}_{\mathrm{H}\cap(\mathrm{F}_n\cap\mathrm{B})^\perp} e$$

since $e \notin \mathrm{F}_n$. Also,

$$\begin{aligned}\bigcap \big(\mathrm{H} \cap (\mathrm{F}_n \cap \mathrm{B})^\perp\big) &= \mathrm{H} \cap \big(\bigcup \mathrm{F}_n \cap \mathrm{B}\big)^\perp = \mathrm{H} \cap (\mathrm{B} \setminus \{e\})^\perp \\ &= \mathrm{H} \cap ([e] + [\mathrm{E} \setminus \mathrm{B}]) = \mathrm{H} \cap [e]\end{aligned}$$

since $\mathrm{H} \subseteq [\mathrm{B}]$, whence $\mathrm{H} \cap (\mathrm{F}_n \cap \mathrm{B})^\perp \xrightarrow{\mathrm{SOT}} \mathrm{H} \cap [e] = 0$ if $e \notin \mathrm{H}$. □

Proof of Theorem 7.2. — Fix $e \in \mathrm{E}$. Let $\mathscr{A}_1 := \{\mathrm{B} \in 2^\mathrm{E};\ e \in \mathrm{B}\}$ and $\mathscr{A}_2 := \{\mathrm{B} \in 2^\mathrm{E};\ e \notin \mathrm{H}_\mathrm{B}\}$. We want to show that $\mathbf{P}^\mathrm{H}(\mathscr{A}_1 \cap \mathscr{A}_2) = 0$. Let $\mathrm{E} \setminus \{e\} = \bigcup \mathrm{F}_n$ for increasing finite sets F_n. By (6.5), we have

$$\mathbf{P}^\mathrm{H}\big(\mathscr{A}_1 \mid \mathscr{F}(\mathrm{F}_n)\big)(\mathrm{B}) = \mathbf{P}^{\mathrm{H}_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}}}(\mathscr{A}_1) = \|\mathrm{P}_{\mathrm{H}_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}}} e\|^2\,.$$

Now

$$(\mathbf{7.2})\qquad (\mathrm{H}_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}})_\mathrm{B} = \overline{(\mathrm{H}_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}})_{\emptyset,\mathrm{E}\setminus\mathrm{B}}}$$

by Lemma 7.3. Write $\mathrm{H}_{\mathrm{B}}^{n} := (\mathrm{H}_{\mathrm{B}})_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}}$. Combining (7.2) with Corollary 6.4 and Lemma 7.3, we obtain

$$(\mathrm{H}_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}})_{\mathrm{B}} = \overline{(\mathrm{H}_{\emptyset,\mathrm{E}\setminus\mathrm{B}})_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}}} \subseteq \overline{(\mathrm{H}_{\mathrm{B}})_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}}} = \mathrm{H}_{\mathrm{B}}^{n}$$

since F_n is finite. If $\mathrm{B} \in \mathscr{A}_1$, then we may apply Lemma 7.5 to $\mathrm{H}_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}}$ and obtain

$$\mathbf{P}^{\mathrm{H}}\big(\mathscr{A}_1 \mid \mathscr{F}(\mathrm{F}_n)\big)(\mathrm{B}) = \|\mathrm{P}_{\mathrm{H}_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}}} e\|^2 \le \|\mathrm{P}_{(\mathrm{H}_{\mathrm{F}_n\cap\mathrm{B},\mathrm{F}_n\setminus\mathrm{B}})_{\mathrm{B}}} e\|^2 \le \|\mathrm{P}_{\mathrm{H}_{\mathrm{B}}^{n}} e\|^2 ,$$

and this tends to 0 as $n \to \infty$ if $\mathrm{B} \in \mathscr{A}_2$ by Lemma 7.7 (applied to $\mathrm{H}_{\mathrm{B}} \subseteq [\mathrm{B}]$). On the other hand, $\mathbf{P}^{\mathrm{H}}\big(\mathscr{A}_1 \mid \mathscr{F}(\mathrm{F}_n)\big) \to \mathbf{P}^{\mathrm{H}}\big(\mathscr{A}_1 \mid \mathscr{F}(\mathrm{E}\setminus\{e\})\big)$ a.s. by Lévy's martingale convergence theorem. Therefore, $\mathbf{P}^{\mathrm{H}}\big(\mathscr{A}_1 \mid \mathscr{F}(\mathrm{E}\setminus\{e\})\big) = 0$ a.s. on the event $\mathscr{A}_1 \cap \mathscr{A}_2$. Write $\mathscr{A}_3 := \big\{\mathrm{B}\,;\ \mathbf{P}^{\mathrm{H}}\big(\mathscr{A}_1 \mid \mathscr{F}(\mathrm{E}\setminus\{e\})\big)(\mathrm{B}) = 0\big\}$, an event that lies in $\mathscr{F}(\mathrm{E}\setminus\{e\})$ and contains $\mathscr{A}_1 \cap \mathscr{A}_2$. We have

$$\begin{aligned}\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \cap \mathscr{A}_2) \le \mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \cap \mathscr{A}_3) &= \mathbf{E}^{\mathrm{H}}\Big[\mathbf{P}^{\mathrm{H}}\big(\mathscr{A}_1 \cap \mathscr{A}_3 \mid \mathscr{F}(\mathrm{E}\setminus\{e\})\big)\Big]\\ &= \mathbf{E}^{\mathrm{H}}\Big[\mathbf{P}^{\mathrm{H}}\big(\mathscr{A}_1 \mid \mathscr{F}(\mathrm{E}\setminus\{e\})\big)\mathbf{1}_{\mathscr{A}_3}\Big] = 0\,.\end{aligned}$$

Since this holds for each $e \in \mathrm{E}$ and E is countable, the theorem follows. □

We now give some applications of Theorem 7.2. For the special case of $\mathrm{H} := \diamondsuit(\mathrm{G})^{\perp}$, we get the following result.

Corollary **7.8.** — *Let* (G, w) *be a network. For the free spanning forest* $\mathfrak{F}$, *we have a.s.* $(\diamondsuit(\mathrm{G})^{\perp})_{\mathfrak{F}} = [\mathfrak{F}]$.

This is a nontrivial theorem about the free spanning forest, different from the trivial statement that the free spanning forest of the free spanning forest is itself, since $(\diamondsuit(\mathrm{G})^{\perp})_{\mathfrak{F}} \subsetneq \diamondsuit(\mathfrak{F})^{\perp}$ except in degenerate cases. The dual statement is easier to interpret:

Corollary **7.9.** — *Let* (G, w) *be a network. For the free spanning forest* $\mathfrak{F}$, *we have that a.s. the free spanning forest of the contracted graph* $\mathrm{G}/\mathfrak{F}$ *is concentrated on the empty set.*

Note that $\mathrm{G}/\mathfrak{F}$ may have vertices of infinite degree.

Proof. — By duality, the free spanning forest has the law of $\mathsf{E}\setminus\mathfrak{B}$ when $\mathfrak{B}$ has the law of $\mathbf{P}^{\diamondsuit(\mathrm{G})}$. We can naturally identify $(\diamondsuit(\mathrm{G}))_{\mathfrak{B}}$ with $\diamondsuit(\mathrm{G}/\mathfrak{B})$. Thus, the fact that $(\diamondsuit(\mathrm{G}))_{\mathfrak{B}} = [\mathfrak{B}]$ $\mathbf{P}^{\diamondsuit(\mathrm{G})}$-a.s. gives the statement of the corollary by duality again. □

We next give a dual form of Theorem 7.2, a form that is a very natural property for infinite matroids with respect to a probability measure $\mathbf{P}^{\mathrm{H}}$. Then we give some applications of this reformulation. The duality that we now use is the following.

Lemma **7.10.** — *Let* H_1 *and* H_2 *be any two closed subspaces of a Hilbert space with corresponding orthogonal projections* P_1, P_2 *and coprojections* $P_1^\perp, P_2^\perp$. *Then* $\overline{P_1 H_2} = H_1$ *iff* $\overline{P_2^\perp H_1^\perp} = H_2^\perp$.

Proof. — By symmetry, it suffices to show that if $\overline{P_1 H_2} \neq H_1$, then $\overline{P_2^\perp H_1^\perp} \neq H_2^\perp$. Now if $\overline{P_1 H_2} \neq H_1$, then there exists a non-0 vector $u \in H_1 \cap (P_1 H_2)^\perp$. Fix such a u. For all $v \in H_2$, we have $0 = (u, P_1 v) = (P_1 u, v) = (u, v)$, so that $u \in H_2^\perp$. Therefore, for any $w \in H_1^\perp$, we have $(u, P_2^\perp w) = (P_2^\perp u, w) = (u, w) = 0$, so that $u \perp P_2^\perp(H_1^\perp)$. Hence, $u \in H_2^\perp \cap \big(P_2^\perp(H_1^\perp)\big)^\perp$, so that $\overline{P_2^\perp H_1^\perp} \neq H_2^\perp$. □

Recall that when H is finite dimensional, $\mathbf{P}^H$ is supported by those subsets $B \subseteq E$ that project to a basis of H under P_H. (Strictly speaking, we have shown this only when E is finite. However, the definition shows that it is true when E is infinite as well, provided H is still finite-dimensional.) The following theorem extends this to the infinite setting insofar as a basis is a spanning set. (The other half of being a basis, minimality, does not hold in general, even for the wired spanning forest of a tree, as shown by the examples in Heicklen and Lyons (2003).)

Theorem **7.11.** — *For any closed subspace* $H \subseteq \ell^2(E)$, *we have* $\overline{[P_H \mathfrak{B}]} = H$ $\mathbf{P}^H$*-a.s.*

Proof. — According to Lemma 7.10, $\overline{[P_H B]} = H$ is equivalent to $(H^\perp)_{[E \setminus B]} = [E \setminus B]$. If we apply Theorem 7.2 to $H^\perp$ and rewrite the conclusion by using (5.6), we see that this holds for $\mathbf{P}^H$-a.e. B. □

Remark **7.12.** — This reasoning shows that Theorem 7.2 could be deduced from an alternative proof of Theorem 7.11. Thus, it would be especially worthwhile to find a simple direct proof of Theorem 7.11.

Our first application of Theorem 7.11 is for $E = \mathbf{Z}$. Let $\mathbf{T} := \mathbf{R}/\mathbf{Z}$ be the unit circle equipped with unit Lebesgue measure. For a measurable function $f : \mathbf{T} \to \mathbf{C}$ and an integer n, the **Fourier coefficient** of f at n is

$$\widehat{f}(n) := \int_{\mathbf{T}} f(t) e^{-2\pi i n t}\, dt\,.$$

If $A \subseteq \mathbf{T}$ is measurable, recall that $S \subseteq \mathbf{Z}$ is **complete for** A if the set $\{f \mathbf{1}_A\,;\ f \in L^2(\mathbf{T}), \widehat{f}\restriction(\mathbf{Z} \setminus S) \equiv 0\}$ is dense in $L^2(A)$. (Again, $L^2(A)$ denotes the set of functions in $L^2(\mathbf{T})$ that vanish outside of A and $\widehat{f}\restriction S$ denotes the restriction of $\widehat{f}$ to S.) The case where A is an interval is quite classical; see, e.g., Redheffer (1977) for a review. A crucial role in that case is played by the following notion of density of S.

Definition **7.13.** — *For an interval* $[a, b] \subset \mathbf{Z} \setminus \{0\}$, *define its* ***aspect***

$$\alpha([a, b]) := \max\{|a|, |b|\} / \min\{|a|, |b|\}\,.$$

For $S \subseteq \mathbf{Z}$*, the* ***Beurling-Malliavin density*** *of* S*, denoted* $\mathsf{BM}(S)$*, is the supremum of those* $D \geq 0$ *for which there exist disjoint nonempty intervals* $I_n \subset \mathbf{Z} \setminus \{0\}$ *with* $|S \cap I_n| \geq D|I_n|$ *for all* n *and* $\sum_{n\geq 1}[\alpha(I_n) - 1]^2 = \infty$.

A simpler form of the Beurling-Malliavin density was provided by Redheffer (1972), who showed that

$$\mathsf{BM}(S) = \inf\left\{c\,;\,\exists \text{ an injection } \beta : S \to \mathbf{Z} \text{ with} \sum_{k\in S}\left|\frac{1}{k} - \frac{c}{\beta(k)}\right| < \infty\right\}. \tag{7.3}$$

Corollary **7.14.** — *Let* $A \subset \mathbf{T}$ *be Lebesgue measurable with measure* $|A|$*. Then there is a set of Beurling-Malliavin density* $|A|$ *in* $\mathbf{Z}$ *that is complete for* A*. Indeed, let* $\mathbf{P}^A$ *be the determinantal probability measure on* $2^{\mathbf{Z}}$ *corresponding to the Toeplitz matrix* $(j, k) \mapsto \widehat{\mathbf{1}_A}(k-j)$*. Then* $\mathbf{P}^A$*-a.e.* $S \subset \mathbf{Z}$ *is complete for* A *and has* $\mathsf{BM}(S) = |A|$.

When A is an interval, the celebrated theorem of Beurling and Malliavan (1967) says that if S is complete for A, then $\mathsf{BM}(S) \geq |A|$. (This holds for S that are not necessarily sets of integers, but we are concerned only with $S \subseteq \mathbf{Z}$.) A. Ulanovskii has pointed out to the author that this inequality can fail dramatically for *certain* subsets $A \subset \mathbf{T}$. We wonder how small $\mathsf{BM}(S)$ can be for general A and $S \subseteq \mathbf{Z}$ that is complete for A.

Proof. — We shall apply Theorem 7.11 with $E := \mathbf{Z}$. We use the Fourier isomorphism $f \mapsto \widehat{f}$ between $L^2(\mathbf{T})$ and $\ell^2(\mathbf{Z})$. Let H be the image of $L^2(A)$ under this isomorphism. Calculation shows that $\mathbf{P}^A = \mathbf{P}^H$ and that $\mathbf{P}^A[e \in S] = |A|$ for all $e \in \mathbf{Z}$ when S has law $\mathbf{P}^A$. The equation $\overline{[P_H S]} = H$ is precisely the statement that S is complete for A. Thus, Theorem 7.11 tells us that $\mathbf{P}^A$-a.e. S is complete for A. It remains to show that $\mathsf{BM}(S) = |A|$ $\mathbf{P}^A$-a.s. If the events $\{e \in S\}$ were independent for $e \in \mathbf{Z}$, this would be a special case of a theorem of Seip and Ulanovskii (1997). It is easy to check that the negative association of $\mathbf{P}^A$ (Theorem 6.5) allows the proof of Seip and Ulanovskii (1997) to carry through to our situation, just as it does for Corollary 6.7. In fact, here is a much shorter proof. First, the ergodic theorem guarantees that the ordinary density of S is $|A|$ for $\mathbf{P}^A$-a.e. S. Thus, $\mathsf{BM}(S) \geq |A|$ $\mathbf{P}^A$-a.s. For the converse inequality, it suffices by symmetry to consider $S \cap \mathbf{Z}^+$, which we write as the increasing sequence $\langle s_n\,;\ n \geq 1\rangle$. By Corollary 6.7 and the Borel-Cantelli lemma, we have

$$\Big|\big|S \cap [1, k]\big| - \big|A\big|k\Big| \leq \sqrt{k \log k}$$

for all but finitely many k a.s. If we substitute $k := s_n$, then we obtain that

$$\left|\frac{1}{s_n} - \frac{|A|}{n}\right| \leq \frac{\sqrt{s_n \log s_n}}{n s_n} = \frac{\sqrt{\log s_n}}{n\sqrt{s_n}}$$

for all but finitely many n a.s. Since $s_n \sim n/|A|$ a.s., it follows that

$$\sum_n \left| \frac{1}{s_n} - \frac{|A|}{n} \right| < \infty \quad \text{a.s.}$$

Thus, $S \cap \mathbf{Z}^+$ a.s. satisfies (7.3) with $\beta(s_n) := n$. □

The dual form of Corollary 7.14 that results from Theorem 7.2 is that the restriction of the Fourier coefficients of f to S for $f \in L^2(A)$ is dense in $\ell^2(S)$ for $\mathbf{P}^A$-a.e. S. This is an equivalent form of Corollary 1.4.

Two more applications of Theorem 7.11 give results for the wired and free spanning forests. However, we do not know their significance. Possibly the one for the FSF, namely,

$$\overline{\left[P_{\diamondsuit}^{\perp}\mathfrak{F}\right]} = \diamondsuit^{\perp} \quad \mathsf{FSF}\text{-a.s.}\,,$$

could be used to glean more information about the FSF, since it is a statement about how large $\mathfrak{F}$ must be.

We next define and prove tail triviality of all measures $\mathbf{P}^H$. For a set $K \subseteq E$, recall that $\mathscr{F}(K)$ denotes the σ-field of events that are measurable with respect to K. Define the **tail** σ-field to be the intersection of $\mathscr{F}(E \setminus K)$ over all finite K. We say that a measure $\mathbf{P}$ on 2^E has **trivial tail** if every event in the tail σ-field has measure either 0 or 1. Recall that tail triviality is equivalent to

$$(\mathbf{7.4})\qquad \begin{gathered} \forall \mathscr{A}_1 \in \mathscr{F}(E) \;\; \forall \epsilon > 0 \;\; \exists K \text{ finite } \;\; \forall \mathscr{A}_2 \in \mathscr{F}(E \setminus K) \\ \left| \mathbf{P}(\mathscr{A}_1 \cap \mathscr{A}_2) - \mathbf{P}(\mathscr{A}_1)\mathbf{P}(\mathscr{A}_2) \right| < \epsilon\,. \end{gathered}$$

(See, e.g., Georgii (1988), p. 120.)

Theorem **7.15.** — *The measure* $\mathbf{P}^H$ *has trivial tail.*

Our proof is modelled on the quantitative proof of tail triviality of FSF and WSF in BLPS (2001). We explain what changes are needed to make that proof work here (and, along the way, give slight corrections). The quantitative form of tail triviality that we prove is this:

Theorem **7.16.** — *Let* F *and* K *be disjoint nonempty subsets of* E *with* K *finite. Let* C *be a subset of* K. *Then*

$$(\mathbf{7.5})\qquad \mathrm{Var}^H\big(\mathbf{P}^H[\mathfrak{B} \cap K = C \mid \mathscr{F}(F)]\big) \le |K| \sum_{e \in K} \|P_{[F]} P_H(e)\|^2\,.$$

If $\mathscr{A}_1 \in \mathscr{F}(K)$ *and* $\mathscr{A}_2 \in \mathscr{F}(F)$, *then*

$$(\mathbf{7.6})\qquad \left| \mathbf{P}^H(\mathscr{A}_1 \cap \mathscr{A}_2) - \mathbf{P}^H(\mathscr{A}_1)\mathbf{P}^H(\mathscr{A}_2) \right| \le \left(2^{|K|}|K| \sum_{e \in K} \|P_{[F]} P_H(e)\|^2 \right)^{1/2}.$$

Before proving Theorem 7.16, we explain why it implies Theorem 7.15. In fact, we show the more quantitative (7.4). Let $\mathscr{A}$ be any event and $\epsilon > 0$. Find a finite set K_1 and $\mathscr{A}_1 \in \mathscr{F}(K_1)$ such that $\mathbf{P}^H(\mathscr{A}_1 \triangle \mathscr{A}) < \epsilon/3$. Now find a finite set K_2 so that

$$\left(2^{|K_1|}|K_1| \sum_{e\in K_1} \|P_{[E\setminus K_2]} P_H(e)\|^2\right)^{1/2} < \epsilon/3\,.$$

Then for all $\mathscr{A}_2 \in \mathscr{F}(E \setminus K_2)$, we have $|\mathbf{P}^H(\mathscr{A} \cap \mathscr{A}_2) - \mathbf{P}^H(\mathscr{A})\mathbf{P}^H(\mathscr{A}_2)| < \epsilon$.

To prove Theorem 7.16, we need to establish some lemmas. Note that both sides of (7.5), as well as of (7.6), are continuous for an increasing sequence of sets F, whence it suffices to prove both inequalities only for F finite. Thus, we may actually assume that E is finite. Assume now that E is finite. Let Q_F be the orthogonal projection onto the (random) subspace $H^F_{F\cap\mathfrak{B}}$ defined in (6.15).

Lemma **7.17.** — *Let* $F \subset E$. *Then*

$$\mathbf{E}^H Q_F = \sum_{S\subseteq F} \mathbf{P}^H[\mathfrak{B} \cap F = S] P_{H^F_S} = P^\perp_{[F]} P_H P^\perp_{[F]}\,. \tag{7.7}$$

The proof is the same as that of Lemma 8.5 of BLPS (2001), where we now establish that for any $e, e' \in E$, we have

$$\mathbf{E}^H(Q_F e,\ e') = \left(P^\perp_{[F]} P_H P^\perp_{[F]} e,\ e'\right).$$

This uses Proposition 6.8 in place of the direct arguments in BLPS (2001).

The next lemma has a precisely parallel proof to that of Lemma 8.6 of BLPS (2001).

Lemma **7.18.** — *Let* $F \subset E$ *and* $u \in \ell^2(E)$. *Then*

$$\mathrm{Var}^H(Q_F u) := \mathbf{E}^H\left[\|Q_F u - \mathbf{E}^H Q_F u\|^2\right] = \left\|P_{[F]} P_H P^\perp_{[F]} u\right\|^2.$$

Proof of Theorem 7.16. — Define $\widetilde{Q}_{F,B}$ to be the orthogonal projection on the subspace $H_{F\cap B, F\setminus B}$. By Theorem 5.1 and (6.5), we have

$$\mathbf{P}^H[\mathfrak{B} \cap K = C \mid \mathscr{F}(F)](B) = \det M^K_{C,B}\,,$$

where

$$M^K_{C,B} := \left[\left(\widetilde{Q}^{C,e}_{F,B} e,\ e'\right)\right]_{e,e'\in K}$$

with notation as follows: For a set C and operator P, write

$$P^{C,e} := \begin{cases} P & \text{if } e \in C, \\ \mathrm{id} - P & \text{if } e \notin C. \end{cases} \tag{7.8}$$

Since $\mathrm{K} \cap \mathrm{F} = \emptyset$, we have that $\big(\widetilde{\mathrm{Q}}^{\mathrm{C},e}_{\mathrm{F},\mathrm{B}} e,\ e'\big) = \big(\mathrm{Q}^{\mathrm{C},e}_{\mathrm{F}} e,\ e'\big)$ for $e, e' \in \mathrm{K}$ on the event that $\mathfrak{B} = \mathrm{B}$. Thus

$$\begin{aligned}
\mathbf{E}^{\mathrm{H}} \mathrm{M}^{\mathrm{K}}_{\mathrm{C},\mathfrak{B}} &= \big[\big(\mathbf{E}^{\mathrm{H}} \mathrm{Q}^{\mathrm{C},e}_{\mathrm{F}} e,\ e'\big)\big]_{e,e' \in \mathrm{K}} \\
&= \Big[\big((\mathrm{P}^{\perp}_{[\mathrm{F}]} \mathrm{P}_{\mathrm{H}} \mathrm{P}^{\perp}_{[\mathrm{F}]})^{\mathrm{C},e} e,\ e'\big)\Big]_{e,e' \in \mathrm{K}} && \text{by (7.7)} \\
&= \Big[\big(\mathrm{P}^{\mathrm{C},e}_{\mathrm{H}} e,\ e'\big)\Big]_{e,e' \in \mathrm{K}} && \text{since } \mathrm{K} \cap \mathrm{F} = \emptyset.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{E}^{\mathrm{H}} \det \mathrm{M}^{\mathrm{K}}_{\mathrm{C},\mathfrak{B}} &= \mathbf{E}^{\mathrm{H}} \mathbf{P}^{\mathrm{H}}[\mathfrak{B} \cap \mathrm{K} = \mathrm{C} \mid \mathscr{F}(\mathrm{F})] \\
&= \mathbf{P}^{\mathrm{H}}[\mathfrak{B} \cap \mathrm{K} = \mathrm{C}] = \det \mathbf{E}^{\mathrm{H}} \mathrm{M}^{\mathrm{K}}_{\mathrm{C},\mathfrak{B}}
\end{aligned}$$

by Theorem 5.1. Furthermore, for any orthogonal projection P, we have

$$\sum_{e' \in \mathrm{K}} |(\mathrm{P}e,\ e')|^2 \le \|\mathrm{P}e\|^2 \le 1$$

because $\langle e';\ e' \in \mathrm{E} \rangle$ is an orthonormal basis for $\ell^2(\mathrm{E})$. Thus, we may apply Lemma 8.7 of BLPS (2001) to obtain

$$\begin{aligned}
\mathrm{Var}^{\mathrm{H}}\big(\mathbf{P}^{\mathrm{H}}[\mathfrak{B} \cap \mathrm{K} = \mathrm{C} \mid \mathscr{F}(\mathrm{F})]\big) &= \mathrm{Var}^{\mathrm{H}}\left(\det \mathrm{M}^{\mathrm{K}}_{\mathrm{C},\mathfrak{B}}\right) \\
&\le |\mathrm{K}| \sum_{e,e' \in \mathrm{K}} \mathrm{Var}^{\mathrm{H}}\big(\mathrm{Q}^{\mathrm{C},e}_{\mathrm{F}} e,\ e'\big) \le |\mathrm{K}| \sum_{e \in \mathrm{K}, e' \in \mathrm{E}} \mathrm{Var}^{\mathrm{H}}\big(\mathrm{Q}^{\mathrm{C},e}_{\mathrm{F}} e,\ e'\big) \\
&= |\mathrm{K}| \sum_{e \in \mathrm{K}} \mathrm{Var}^{\mathrm{H}}\big(\mathrm{Q}^{\mathrm{C},e}_{\mathrm{F}} e\big) = |\mathrm{K}| \sum_{e \in \mathrm{K}} \mathrm{Var}^{\mathrm{H}}\big(\mathrm{Q}_{\mathrm{F}} e\big) \\
&= |\mathrm{K}| \sum_{e \in \mathrm{K}} \|\mathrm{P}_{[\mathrm{F}]} \mathrm{P}_{\mathrm{H}}(e)\|^2 ,
\end{aligned}$$

using Lemma 7.18. This proves (7.5).

To deduce (7.6) from (7.5), write $a := 2^{|\mathrm{K}|} |\mathrm{K}| \sum_{e \in \mathrm{K}} \|\mathrm{P}_{[\mathrm{F}]} \mathrm{P}_{\mathrm{H}}(e)\|^2$. Then for all $\mathscr{A}_1 \in \mathscr{F}(\mathrm{K})$, we have

$$\mathrm{Var}^{\mathrm{H}}\big(\mathbf{P}(\mathscr{A}_1 \mid \mathscr{F}(\mathrm{F}))\big) \le a$$

since $\mathscr{A}_1$ is the union of at most $2^{|\mathrm{K}|}$ disjoint cylinder events of the form $\{\mathfrak{B} \cap \mathrm{K} = \mathrm{C}\}$. Therefore for all $\mathscr{A}_2 \in \mathscr{F}(\mathrm{F})$,

$$\Big|\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \mid \mathscr{A}_2) - \mathbf{P}^{\mathrm{H}}(\mathscr{A}_1)\Big|^2 \mathbf{P}^{\mathrm{H}}(\mathscr{A}_2) \le a ,$$

so that

$$\Big|\mathbf{P}^{\mathrm{H}}(\mathscr{A}_1 \cap \mathscr{A}_2) - \mathbf{P}^{\mathrm{H}}(\mathscr{A}_1)\mathbf{P}^{\mathrm{H}}(\mathscr{A}_2)\Big|^2 \le a\mathbf{P}^{\mathrm{H}}(\mathscr{A}_2) \le a .$$

This is the same as (7.6). □

8. Positive contractions

We have seen that the matrix of any orthogonal projection gives a determinantal probability measure. We now do the same for positive contractions and give their properties.

We call Q a **positive contraction** if Q is a self-adjoint operator on $\ell^2(\mathrm{E})$ such that for all $u \in \ell^2(\mathrm{E})$, we have $0 \le (\mathrm{Q}u, u) \le (u, u)$. To show existence of a corresponding determinantal probability measure, which we shall denote $\mathbf{P}^{\mathrm{Q}}$, let P_{H} be any orthogonal projection that is a **dilation** of Q, i.e., H is a closed subspace of $\ell^2(\mathrm{E}')$ for some $\mathrm{E}' \supseteq \mathrm{E}$ and for all $u \in \ell^2(\mathrm{E})$, we have $\mathrm{Q}u = \mathrm{P}_{\ell^2(\mathrm{E})}\mathrm{P}_{\mathrm{H}}u$, where we regard $\ell^2(\mathrm{E}')$ as the orthogonal sum $\ell^2(\mathrm{E}) \oplus \ell^2(\mathrm{E}' \setminus \mathrm{E})$. (In this case, Q is also called the **compression** of P_{H} to $\ell^2(\mathrm{E})$.) The existence of a dilation is standard and is easily constructed: Let E′ be the union of E with a disjoint copy $\hat{\mathrm{E}}$ of E. Let T be the positive square root of Q and let $\hat{\mathrm{T}}$ be the positive square root of $\mathrm{I} - \mathrm{Q}$. The operator whose block matrix is

$$\begin{pmatrix} \mathrm{Q} & \mathrm{T}\hat{\mathrm{T}} \\ \mathrm{T}\hat{\mathrm{T}} & \mathrm{I} - \mathrm{Q} \end{pmatrix}$$

is easily checked to be self-adjoint and idempotent, hence it is an orthogonal projection onto a closed subspace H. Having chosen a dilation, we simply define $\mathbf{P}^{\mathrm{Q}}$ as the law of $\mathfrak{B} \cap \mathrm{E}$ when $\mathfrak{B}$ has the law $\mathbf{P}^{\mathrm{H}}$. Then (1.1) is a special case of (5.3).

Of course, when Q is the orthogonal projection onto a subspace H, then $\mathbf{P}^{\mathrm{Q}} = \mathbf{P}^{\mathrm{H}}$.

The basic properties of $\mathbf{P}^{\mathrm{Q}}$ follow from those for orthogonal projections. In the following, we write $\mathrm{Q}_1 \le \mathrm{Q}_2$ if $(\mathrm{Q}_1 u, u) \le (\mathrm{Q}_2 u, u)$ for all $u \in \ell^2(\mathrm{E})$.

Theorem **8.1.** — *Let* Q *be a positive contraction. For any finite* $\mathrm{A}, \mathrm{B} \subseteq \mathrm{E}$, *we have*

$$\mathbf{P}^{\mathrm{Q}}[\mathrm{A} \subseteq \mathfrak{S}, \mathrm{B} \cap \mathfrak{S} = \emptyset] = \left(\bigwedge_{e \in \mathrm{A}} \mathrm{Q}e \wedge \bigwedge_{e \in \mathrm{B}} (\mathrm{I} - \mathrm{Q})e,\ \theta_{\mathrm{A}} \wedge \theta_{\mathrm{B}} \right). \tag{8.1}$$

The measure $\mathbf{P}^{\mathrm{Q}}$ *has conditional negative associations (with external fields) and a trivial tail* σ*-field. If* $\mathrm{A} \subseteq \mathrm{E}$ *is finite and* $\mu := \mathbf{E}^{\mathrm{Q}}[|\mathfrak{S} \cap \mathrm{A}|]$, *then for any* $a > 0$, *we have*

$$\mathbf{P}^{\mathrm{Q}}\left[\Big| |\mathfrak{S} \cap \mathrm{A}| - \mu \Big| \ge a \right] \le 2e^{-2a^2/|\mathrm{A}|}. \tag{8.2}$$

If Q_1 *and* Q_2 *are commuting positive contractions and* $\mathrm{Q}_1 \le \mathrm{Q}_2$, *then* $\mathbf{P}^{\mathrm{Q}_1} \preccurlyeq \mathbf{P}^{\mathrm{Q}_2}$.

Remark **8.2.** — Independently, Shirai and Takahashi (2003) showed that $\mathbf{P}^{\mathrm{Q}}$ has a trivial tail σ-field when the spectrum of Q lies in (0, 1).

Proof. — The first four properties are immediate consequences of (5.2), Theorem 6.5, Theorem 7.15, and Corollary 6.7. (Of course, many other properties follow from the negative association; we mention (8.2) merely as an example.) The last statement will follow from Theorem 7.1 once we show that the hypothesized commutativity implies that we may take dilations P_{H_i} of Q_i with $H_1 \subseteq H_2$. To do this, we use the following form of the spectral theorem: There is a measure space (X, μ), two Borel functions $f_i : X \to [0, 1]$, and a unitary map $U : \ell^2(E) \to L^2(\mu)$ such that $UQ_iU^{-1} : g \mapsto f_i g$ for $i = 1, 2$ and any $g \in L^2(\mu)$ (apply Theorem IX.4.6, p. 272, of Conway (1990) to the normal operator $Q_1 + iQ_2$). Since $Q_1 \le Q_2$, we have $f_1 \le f_2$. Use U to identify $\ell^2(E)$ with $L^2(\mu)$ and to identify Q_i with $M_i := UQ_iU^{-1}$. Let λ denote Lebesgue measure on $[0, 1]$ and define $A_i := \{(x, y)\,;\; x \in X,\ 0 \le y \le f_i(x)\}$. Let H_i be the subspace of functions in $L^2(\mu \otimes \lambda)$ that vanish outside A_i. Since $A_1 \subseteq A_2$, we have $H_1 \subseteq H_2$. Embed $L^2(\mu)$ in $L^2(\mu \otimes \lambda)$ by $g \mapsto g \otimes \mathbf{1}$ and identify $L^2(\mu)$ with its image. Then M_i is the compression of P_{H_i} to $L^2(\mu)$, as desired. □

A formula for the probability measure $\mathbf{P}^Q(\,\bullet \mid A \subseteq \mathfrak{S}, B \cap \mathfrak{S} = \emptyset)$ follows from applying (6.5) to a dilation of Q. However, this is not very explicit. Often conditioning on just $A \subseteq \mathfrak{S}$ is important, so we give the following direct formula for that case. Note that we allow A to be infinite; if $A = \bigcup_n A_n$ with A_n finite, then $\mathbf{P}^Q(\,\bullet \mid A_n \subseteq \mathfrak{S})$ is a stochastically decreasing sequence of probability measures by Theorem 6.5 and so defines $\mathbf{P}^Q(\,\bullet \mid A \subseteq \mathfrak{S})$. We shall write

$$(u,\ v)_Q := (Qu,\ v)$$

for the inner product on $\ell^2(E)$ induced by Q. Let $[E]_Q$ be the completion of $\ell^2(E)$ in this inner product and $P^{\perp}_{[A]_Q}$ be the orthogonal projection in $[E]_Q$ onto the subspace orthogonal to A.

Proposition **8.3.** — *Let* Q *be a positive contraction on* $\ell^2(E)$ *and* $A \subset E$. *When* $\mathfrak{S}$ *has law* $\mathbf{P}^Q$ *conditioned on* $A \subseteq \mathfrak{S}$, *then the law of* $\mathfrak{S} \cap (E \setminus A)$ *is the determinantal probability measure corresponding to the positive contraction on* $\ell^2(E \setminus A)$ *whose* (e, f)*-matrix entry is*

$$\left(P^{\perp}_{[A]_Q} e,\ P^{\perp}_{[A]_Q} f\right)_Q .$$

An equivalent expression was found independently by Shirai and Takahashi (2002), Corollary 6.5.

Proof. — Because of (6.5), we know that the law of $\mathfrak{S} \cap (E \setminus A)$ is the determinantal probability measure corresponding to the compression of some orthogonal projection, i.e., to some positive contraction. What remains is to show that for any finite $B \subset E \setminus A$, we have

$$\mathbf{P}^Q[B \subseteq \mathfrak{S} \mid A \subseteq \mathfrak{S}] = \det\left[\left(P^{\perp}_{[A]_Q} e,\ P^{\perp}_{[A]_Q} f\right)_Q\right]_{e,f \in B} .$$

By continuity, it suffices to do this when A is finite. Now,

$$\mathbf{P}^{\mathrm{Q}}[\mathrm{B} \subseteq \mathfrak{S} \mid \mathrm{A} \subseteq \mathfrak{S}] = \frac{\det[(\mathrm{Q}e, f)]_{e,f \in \mathrm{A} \cup \mathrm{B}}}{\det[(\mathrm{Q}e, f)]_{e,f \in \mathrm{A}}} = \frac{\det[(e, f)_{\mathrm{Q}}]_{e,f \in \mathrm{A} \cup \mathrm{B}}}{\det[(e, f)_{\mathrm{Q}}]_{e,f \in \mathrm{A}}} = \frac{\|\theta_{\mathrm{B}} \wedge \theta_{\mathrm{A}}\|_{\mathrm{Q}}^2}{\|\theta_{\mathrm{A}}\|_{\mathrm{Q}}^2}.$$

We use the following fact about exterior algebras. For any vectors $u_1, u_2, \ldots, u_m$, $v_1, v_2, \ldots, v_n$ with H defined to be the span of $v_1, \ldots, v_n$, we have

$$\bigwedge_{i=1}^{m} u_i \wedge \bigwedge_{j=1}^{n} v_j = \bigwedge_{i=1}^{m} \mathrm{P}_{\mathrm{H}}^{\perp} u_i \wedge \bigwedge_{j=1}^{n} v_j .$$

This is because $\mathrm{P}_{\mathrm{H}} u_i \wedge \bigwedge_{j=1}^{n} v_j = 0$. Thus,

$$\mathbf{P}^{\mathrm{Q}}[\mathrm{B} \subseteq \mathfrak{S} \mid \mathrm{A} \subseteq \mathfrak{S}] = \frac{\left\| \bigwedge_{e \in \mathrm{B}} \mathrm{P}_{[\mathrm{A}]_{\mathrm{Q}}}^{\perp} e \wedge \theta_{\mathrm{A}} \right\|_{\mathrm{Q}}^2}{\|\theta_{\mathrm{A}}\|_{\mathrm{Q}}^2} = \Big\| \bigwedge_{e \in \mathrm{B}} \mathrm{P}_{[\mathrm{A}]_{\mathrm{Q}}}^{\perp} e \Big\|_{\mathrm{Q}}^2 ,$$

as desired. □

Remark **8.4.** — If (1.1) is given, then (8.1) can be deduced from (1.1) without using our general theory and, in fact, without assuming that the matrix Q is self-adjoint. Indeed, suppose that X is any diagonal matrix. Denote its (e, e)-entry by X_e. Comparing coefficients of X_e shows that (1.1) implies

$$\mathbf{E}\Big[\prod_{e \in \mathrm{A}} \big(\mathbf{1}_{\{e \in \mathfrak{S}\}} + \mathrm{X}_e\big)\Big] = \det\big((\mathrm{Q} + \mathrm{X}) \restriction \mathrm{A}\big) .$$

If A is partitioned as $\mathrm{A}_1 \cup \mathrm{A}_2$ and we choose X so that $\mathrm{X}_e = -\mathbf{1}_{\mathrm{A}_2}(e)$, then we obtain

$$\mathbf{P}[\mathrm{A}_1 \subseteq \mathfrak{S}, \mathrm{A}_2 \cap \mathfrak{S} = \emptyset] = \det\big(\mathrm{Q}^{\mathrm{A}_2} \restriction (\mathrm{A}_1 \cup \mathrm{A}_2)\big) , \tag{8.3}$$

where Q^{A} denotes the matrix whose rows are the same as those of Q except for those rows indexed by $e \in \mathrm{A}$, which instead equals that row of Q subtracted from the corresponding row of the identity matrix. In other words, if $\mathrm{Q}_{e,f}$ denotes the (e,f)-entry of Q, then the (e,f)-entry of Q^{A} is

$$\mathbf{1}_{\mathrm{A}}(e) + (-1)^{\mathbf{1}_{\mathrm{A}}(e)} \mathrm{Q}_{e,f} .$$

This is another form of (8.1). (Equation (8.3) amounts to an explicit form of the inclusion-exclusion principle for a determinantal probability measure.)

Remark **8.5.** — If Q is a self-adjoint matrix such that (1.1) defines a probability measure, then necessarily Q is a positive contraction. The fact that $\mathrm{Q} \geq 0$ is a consequence of having nonnegative minors, while $\mathrm{I} - \mathrm{Q} \geq 0$ follows from observing that $\mathrm{I} - \mathrm{Q}$ also defines a determinantal probability measure, the dual to the one defined by Q.

9. Open questions: General theory

Our last sections present open questions organized by topic. The first two sections concern general determinantal probability measures, while the others examine specific types of measures.

In order to extend (6.5) to the case where A and B may be infinite and thereby obtain a version of conditional probabilities, define

$$\mathrm{H}^*_{\mathrm{A,B}} := \overline{(\mathrm{H} \cap \mathrm{A}^\perp) + [\mathrm{A} \cup \mathrm{B}]} \cap \mathrm{B}^\perp$$

and

$$\mathrm{H}^{**}_{\mathrm{A,B}} = \left(\overline{\mathrm{H} + [\mathrm{B}]} \cap (\mathrm{A} \cup \mathrm{B})^\perp\right) + [\mathrm{A}]\,.$$

One can show that $\mathrm{H}^*_{\mathrm{A,B}} \subseteq \mathrm{H}^{**}_{\mathrm{A,B}}$, but that they are not necessarily equal.

The following conjecture would greatly simplify the proof of Theorem 7.2 above.

Conjecture **9.1.** — Let H be a closed subspace of $\ell^2(\mathrm{E})$ and $\mathrm{K} \subset \mathrm{E}$. A version of the conditional probability measure $\mathbf{P}^{\mathrm{H}}$ given $\mathscr{F}(\mathrm{K})$ is $\mathrm{B} \mapsto \mathbf{P}^{\mathrm{H}^*_{\mathrm{K}\cap\mathrm{B},\mathrm{K}\setminus\mathrm{B}}}$ and another is given by $\mathrm{B} \mapsto \mathbf{P}^{\mathrm{H}^{**}_{\mathrm{K}\cap\mathrm{B},\mathrm{K}\setminus\mathrm{B}}}$.

We say that $\mathscr{A}_1, \mathscr{A}_2 \subset 2^{\mathrm{E}}$ **occur disjointly for** $\mathrm{F} \subseteq \mathrm{E}$ if there are disjoint sets $\mathrm{F}_1, \mathrm{F}_2 \subset \mathrm{E}$ such that

$$\{\mathrm{K} \subseteq \mathrm{E}\,;\ \mathrm{K} \cap \mathrm{F}_i = \mathrm{F} \cap \mathrm{F}_i\} \subseteq \mathscr{A}_i$$

for $i = 1, 2$. A probability measure $\mathbf{P}$ on 2^{E} is said to have the **BK property** if

$$\mathbf{P}[\mathscr{A}_1 \text{ and } \mathscr{A}_2 \text{ occur disjointly for } \mathfrak{S}] \le \mathbf{P}[\mathfrak{S} \in \mathscr{A}_1]\mathbf{P}[\mathfrak{S} \in \mathscr{A}_2]$$

for every pair $\mathscr{A}_1, \mathscr{A}_2 \subset 2^{\mathrm{E}}$ of increasing events. Does every determinantal probability measure $\mathbf{P}^{\mathrm{Q}}$ have the BK property? The BK inequality of van den Berg and Kesten (1985) says that this holds when Q is a diagonal matrix, i.e., when $\mathbf{P}$ is product measure. The answer is unknown even in the special case of uniform spanning trees, where it is conjectured to hold in BLPS (2001).

Is entropy concave in Q for fixed E? That is, for finite E and a positive contraction Q, define the **entropy** of $\mathbf{P}^{\mathrm{Q}}$ to be

$$\mathsf{Ent}(\mathrm{Q}) := -\sum_{\mathrm{A}\in 2^{\mathrm{E}}} \mathbf{P}^{\mathrm{Q}}[\mathrm{A}] \log \mathbf{P}^{\mathrm{Q}}[\mathrm{A}]\,.$$

Numerical calculation supports the following conjecture.

Conjecture **9.2.** — For any positive contractions Q_1 and Q_2, we have

$$\mathsf{Ent}\big((\mathrm{Q}_1 + \mathrm{Q}_2)/2\big) \ge \big(\mathsf{Ent}(\mathrm{Q}_1) + \mathsf{Ent}(\mathrm{Q}_2)\big)/2\,. \qquad (\mathbf{9.1})$$

In BLPS (2001), it is asked whether the free and wired spanning forests are mutually singular when they are not equal. One might hope that the following more general statement holds: if $H_1 \subsetneq H_2$, then the corresponding probability measures $\mathbf{P}^{H_1}$ and $\mathbf{P}^{H_2}$ are mutually singular. However, this general statement is false, as shown by Heicklen and Lyons (2003). Nevertheless, since it is true trivially for finite E, it seems likely that there are interesting sufficient conditions for $\mathbf{P}^{H_1}$ and $\mathbf{P}^{H_2}$ to be mutually singular.

If $\mathbf{P}^{H_1} \preccurlyeq \mathbf{P}^{H_2}$, must there exist a subspace $H_3 \subseteq H_2$ such that $\mathbf{P}^{H_1} = \mathbf{P}^{H_3}$? (This was answered in the negative by Lewis Bowen after a preprint was circulated.)

Given the value of Theorem 7.11 and of its dual form Theorem 7.2, it seems desirable to extend other properties of matroids to the infinite setting.

10. Open questions: Coupling

A **coupling** of two probability measures $\mathbf{P}^1$, $\mathbf{P}^2$ on 2^E is a probability measure μ on $2^E \times 2^E$ whose coordinate projections are $\mathbf{P}^1$, $\mathbf{P}^2$, meaning that for all events $\mathscr{A} \subseteq 2^E$, we have

$$\mu\big\{(A_1, A_2)\,;\ A_1 \in \mathscr{A}\big\} = \mathbf{P}^1(\mathscr{A})$$

and

$$\mu\big\{(A_1, A_2)\,;\ A_2 \in \mathscr{A}\big\} = \mathbf{P}^2(\mathscr{A})\,.$$

A coupling μ is called **monotone** if

$$\mu\big\{(A_1, A_2)\,;\ A_1 \subseteq A_2\big\} = 1\,.$$

By Strassen's 1965 theorem, stochastic domination $\mathbf{P}^1 \preccurlyeq \mathbf{P}^2$ is equivalent to the existence of a monotone coupling of $\mathbf{P}^1$ and $\mathbf{P}^2$. A very interesting open question that arises from Theorem 6.2 is to find a natural or explicit monotone coupling of $\mathbf{P}^{H'}$ and $\mathbf{P}^H$ when $H' \subset H$.

A coupling μ is **disjoint** if $\mu\big\{(A_1, A_2)\,;\ A_1 \cap A_2 = \emptyset\big\} = 1$. A coupling μ has **union marginal P** if for all events $\mathscr{A} \subseteq 2^E$, we have

$$\mathbf{P}(\mathscr{A}) = \mu\big\{(A_1, A_2)\,;\ A_1 \cup A_2 \in \mathscr{A}\big\}\,.$$

Question **10.1.** — Given $H = H_1 \oplus H_2$, is there a (natural or otherwise) disjoint coupling of $\mathbf{P}^{H_1}$ and $\mathbf{P}^{H_2}$ with union marginal $\mathbf{P}^H$?

This is easily seen to be the case when $H = \ell^2(E)$: The probability measure μ on $2^E \times 2^E$ defined by

$$\mu\{(A, E \setminus A)\,;\ A \in \mathscr{A}\} := \mathbf{P}^{H_1}(\mathscr{A})$$

and

$$\mu\big\{(A, B)\,;\ B \neq E \setminus A\} := 0$$

does this, as we can see by Corollary 5.3. A positive answer in general to Question 10.1 would give the following more general result (by the method of proof of Theorem 8.1): If Q_1 and Q_2 are commuting positive contractions on $\ell^2(E)$ such that $Q_1 + Q_2 \le I$, then there is a disjoint coupling of $\mathbf{P}^{Q_i}$ with union marginal $\mathbf{P}^{Q_1+Q_2}$. We note that the requirement of being disjoint is superfluous, although useful to keep in mind:

Proposition **10.2.** — *If* Q_1 *and* Q_2 *are positive contractions on* $\ell^2(E)$ *such that* $Q_1 + Q_2 \le I$, *then any coupling of* $\mathbf{P}^{Q_1}$, $\mathbf{P}^{Q_2}$ *with union marginal* $\mathbf{P}^{Q_1+Q_2}$ *is necessarily a disjoint coupling.*

Proof. — Let the coupling be μ, which picks a random pair $(\mathfrak{S}_1, \mathfrak{S}_2) \in 2^E \times 2^E$. Then for all $e \in E$, we have

$$\begin{aligned}\big((Q_1 + Q_2)e,\ e\big) &= \mathbf{P}^{Q_1+Q_2}[e \in \mathfrak{S}] = \mu[e \in \mathfrak{S}_1 \cup \mathfrak{S}_2]\\ &= \mu[e \in \mathfrak{S}_1] + \mu[e \in \mathfrak{S}_2] - \mu[e \in \mathfrak{S}_1 \cap \mathfrak{S}_2]\\ &= \mathbf{P}^{Q_1}[e \in \mathfrak{S}_1] + \mathbf{P}^{Q_2}[e \in \mathfrak{S}_2] - \mu[e \in \mathfrak{S}_1 \cap \mathfrak{S}_2]\\ &= (Q_1 e,\ e) + (Q_2 e,\ e) - \mu[e \in \mathfrak{S}_1 \cap \mathfrak{S}_2]\\ &= \big((Q_1 + Q_2)e,\ e\big) - \mu[e \in \mathfrak{S}_1 \cap \mathfrak{S}_2]\,.\end{aligned}$$

Therefore $\mu[e \in \mathfrak{S}_1 \cap \mathfrak{S}_2] = 0$. Since this holds for each e, we get the result. □

If a natural monotone coupling is found, it ought to provide a coupling of the free and wired spanning forests that is invariant under all automorphisms of the underlying graph, G. This should help in understanding the free spanning forest. (A specific instance is given below.)

We shall give some partial results on the general question.

Proposition **10.3.** — *If* $H \subset H'$ *are two closed subspaces of* $\ell^2(E)$, *then there is a monotone coupling of* $\mathbf{P}^H$ *and* $\mathbf{P}^{H'}$ *concentrated on the set* $\{(B_1, B_2)\,;\ |B_2 \setminus B_1| = k\}$, *where* k *is the codimension of* H *in* H' *(possibly* $k = \infty$*).*

We do not know whether every monotone coupling has this property.

Of course, Proposition 10.3 is trivial when $|E| < \infty$. To prove Proposition 10.3 when E is infinite, we first prove a lemma that shows that in the finite-dimensional codimension-one case, every monotone coupling gives rise to a disjoint coupling with the proper union marginal:

Lemma **10.4.** — *Let* H *be a finite-dimensional subspace of* $\ell^2(E)$ *and* u *be a unit vector in* $H^\perp$. *Let* H' *be the span of* H *and* u. *If* μ *is any monotone coupling of* $\mathbf{P}^H$ *and* $\mathbf{P}^{H'}$, *then for every event* $\mathscr{A}$,

$$\mu\{(B, B')\,;\ B' \setminus B \in \mathscr{A}\} = \mathbf{P}^u(\mathscr{A})\,.$$

Proof. — We have that

$$\mu\{(B, B') ;\ |B| = \dim H\} = 1$$

and

$$\mu\{(B, B') ;\ |B'| = \dim H + 1\} = 1 .$$

Therefore, it suffices to show that for every $e \in E$,

$$\mu\{(B, B') ;\ e \in B' \setminus B\} = \mathbf{P}^u(\{e\}) .$$

This equation holds because the left-hand side is equal to

$$\begin{aligned}\mu\{(B, B') ;\ e \in B'\} - \mu\{(B, B') ;\ e \in B\} &= \mathbf{P}^{H'}[e \in \mathfrak{B}'] - \mathbf{P}^{H}[e \in \mathfrak{B}] \\ &= \|P_{H'}e\|^2 - \|P_H e\|^2 = \|P_{[u]}e\|^2 \\ &= \mathbf{P}^u[\{e\}] .\end{aligned}$$

□

It follows from this and duality considerations that Question 10.1 has a positive answer whenever $|E| \leq 5$. We have tested by computer thousands of random instances of Question 10.1 for $|E| = 6, 7, 8, 9$ and all have a positive answer.

Proof of Proposition 10.3. — The case that H′ is finite dimensional is trivial, so suppose that H′ is infinite dimensional.

Suppose first that $k = 1$. Let $H' = H \oplus [u]$ and choose any increasing sequence of finite-dimensional subspaces H_i $(i \geq 1)$ of H whose union is dense in H. Let μ_i be any monotone coupling of $\mathbf{P}^{H_i}$ and $\mathbf{P}^{H_i \oplus [u]}$ for each $i \geq 1$. Let μ be any weak* limit point of μ_i. Then μ is a monotone coupling of $\mathbf{P}^{H}$ and $\mathbf{P}^{H'}$. Furthermore, since

$$\mu_i\{(B, B') ;\ B' \setminus B = \{e\}\} = \mathbf{P}^u[\{e\}]$$

for each i, the same holds for μ. This gives the desired conclusion.

Now suppose that $1 < k < \infty$. Let $H' = H_1 \supset H_2 \supset H_3 \supset \cdots \supset H_k = H$ be a decreasing sequence of subspaces with H_{i+1} having codimension 1 in H_i for each $i = 1, ..., k-1$. By what we have shown, we may choose monotone couplings μ_i of $\mathbf{P}^{H_{i+1}}$ with $\mathbf{P}^{H_i}$ for each $i = 1, ..., k-1$ with the property that

$$\mu_i\{(B, B') ;\ |B' \setminus B| = 1\} = 1 .$$

Choose (B_i, B_i') with distribution μ_i and independently of each other. Then the distribution μ of (B_{k-1}, B_1') given that $B_i = B_{i+1}'$ for each $i = 1, ..., k-1$ is the desired coupling.

Finally, if $k = \infty$, then choose a decreasing sequence of subspaces $H' = H_1 \supset H_2 \supset H_3 \supset \cdots \supset H_i \supset \cdots$ with H_{i+1} having codimension 1 in H_i for each $i \geq 1$ and $\bigcap H_i = H$. Let μ_i be any monotone coupling of $\mathbf{P}^{H'}$ and $\mathbf{P}^{H_i}$ with

$$\mu_i\{(B, B') ;\ |B' \setminus B| = i\} = 1 .$$

Let μ be any weak* limit point of μ_i. Then μ is the desired coupling. □

An example of the usefulness of coupling is as follows. Let $G = (\mathsf{V}, \mathsf{E})$ be a proper planar graph with planar dual $G^\dagger = (\mathsf{V}^\dagger, \mathsf{E}^\dagger)$. Let H be a subspace of $\ell^2(\mathsf{E})$. Let ϕ be the map $e \mapsto e^\dagger$. Then ϕ induces a map $\ell^2(\mathsf{E}) \to \ell^2(\mathsf{E}^\dagger)$ that sends H to a subspace $H^\dagger$ of $\ell^2(\mathsf{E}^\dagger)$. For example, $\bigstar(G)^\dagger = \diamondsuit(G^\dagger)$ and $\diamondsuit(G)^\dagger = \bigstar(G^\dagger)$. The disjoint coupling of $\mathbf{P}^H$ and $\mathbf{P}^{H^\perp}$ of the first paragraph of this section gives a coupling of $\mathbf{P}^H$ on $\ell^2(\mathsf{E})$ and $\mathbf{P}^{(H^\dagger)^\perp}$ on $\ell^2(\mathsf{E}^\dagger)$ for which exactly one of e and $e^\dagger$ appear in $\mathfrak{B}$ and $\mathfrak{B}^\dagger$ for each $e \in \mathsf{E}$. For example, if $H = \bigstar(G)$, then $\mathbf{P}^H = \mathsf{WSF}(G)$; since $(H^\dagger)^\perp = \diamondsuit(G^\dagger)^\perp$, we obtain the disjoint coupling of $\mathsf{WSF}(G)$ with $\mathsf{FSF}(G^\dagger)$ used in BLPS (2001). This is the just about the only method known to derive much information about FSF when $\mathsf{FSF} \neq \mathsf{WSF}$. Note that if G is not planar but is embedded on a surface, then $\bigstar(G)^\dagger$ is the span of the facial (contractible) cycles of $G^\dagger$, which may be smaller than $\diamondsuit(G^\dagger)$. (This is discussed further in Sect. 12 below.)

11. Open questions: Groups

We shall consider first finite groups, then infinite groups.

Suppose that E is a finite group. Then $\ell^2(E)$ is the group algebra of E. Invariant subspaces H give subrepresentations of the regular representation and give invariant probability measures $\mathbf{P}^H$. There is a canonical decomposition

$$\ell^2(E) = \bigoplus_{j=1}^{s} H_j \,,$$

where each H_j is an invariant subspace containing all isomorphic copies of a given irreducible representation. (See, e.g., Fulton and Harris (1991).) The matrix of P_{H_j} is given by the character χ_{H_j} of the representation, namely, the (e,f)-entry is $\overline{\chi_{H_j}(ef^{-1})}/|E|$ (Fulton and Harris (1991), p. 23, (2.32)). Can we (disjointly) couple all measures $\mathbf{P}^{H_j}$ so that every partial union has marginal equal to $\mathbf{P}^H$ for H the corresponding partial sum? In other words, is there a probability measure μ on $\prod_{j=1}^{s} 2^E$ picking a random s-tuple $\langle \mathfrak{S}_1, \ldots, \mathfrak{S}_s \rangle$ such that for every $J \subseteq \{1, \ldots, s\}$, the law of $\bigcup_{j \in J} \mathfrak{S}_j$ is $\mathbf{P}^{H_J}$, where $H_J := \bigoplus_{j \in J} H_j$? We call such a coupling **complete**.

Consider the case $E = \mathbf{Z}_n$. All irreducible representations are 1-dimensional and there are n of them: for each $k \in \mathbf{Z}_n$, we have the representation

$$m \mapsto e^{2\pi i k m/n} \qquad (m \in \mathbf{Z}_n) \,.$$

Thus, a complete coupling would be a random permutation of $\mathbf{Z}_n$ with special properties. By averaging, we may always assume that any complete coupling is invariant. Do they always exist? If so, the set of invariant complete couplings is a polytope. What are its extreme points or the supports of the extreme points? What is its barycenter? What asymptotic properties distinguish it from the uniform permutation? Testing by computer indicates existence for all $n \le 7$. Thus, it would appear that complete couplings always exist on $\mathbf{Z}_n$.

One should be aware that it is not always possible to completely couple 4 measures from orthogonal subspaces when the subspaces are not invariant, as one can show from the following example. Let $v_1 := \langle 1, 1, -3, 1\rangle$, $v_2 := \langle 1, -1, 5, 2\rangle$, $v_3 := \langle 1, 1, -2, -2\rangle$, and $v_4 := \langle -3, 2, 1, 4\rangle$. Let u_j be the corresponding vectors resulting from the Gram-Schmidt procedure, that is, $u_1 := v_1/\|v_1\|$, $u_2 := P^{\perp}_{[v_1]}v_2/\|P^{\perp}_{[v_1]}v_2\|$, etc. Then if $H_j := [u_j]$, computer calculation shows the impossibility of complete coupling.

Now let G be a Cayley graph of a finitely generated infinite group Γ. Analogies with percolation (see, e.g., Lyons (2000) for a review) and with minimal spanning forests (see Lyons, Peres, and Schramm (2003)) suggest the following possibilities. Let H be a Γ-invariant subspace of $\ell^2(\mathsf{E})$.

- If H is a proper subspace of the star space $\bigstar$ of G, then all connected components of $\mathfrak{B}$ are finite $\mathbf{P}^{H}$-a.s.
- If $\bigstar \subsetneq H \subsetneq \diamondsuit^{\perp}$, where $\diamondsuit$ is the cycle space of G, then $\mathfrak{B}$ has infinitely many (infinite) components $\mathbf{P}^{H}$-a.s.
- If $\diamondsuit^{\perp} \subsetneq H$, then $\mathfrak{B}$ has a single (infinite) component $\mathbf{P}^{H}$-a.s.

The second statement is shown to be true by Lyons (2003). We do not know whether the others are true. However, when $H \subsetneq \bigstar$, the expected degree of a vertex with respect to $\mathbf{P}^{H}$ is less than 2, its expected degree in the WSF (BLPS (2001)). By Theorem 6.1 of Benjamini, Lyons, Peres, and Schramm (1999), it follows that $\mathfrak{B}$ has infinitely many finite components $\mathbf{P}^{H}$-a.s. It is shown in Lyons (2003) that the last bulleted statement above implies a positive answer to an important question of Gaboriau (2002), showing that the cost of Γ is equal to 1 plus the first ℓ^2-Betti number of Γ.

12. Open questions: Surface graphs and CW-complexes

For graphs $G = (\mathsf{V}, \mathsf{E})$, one need not restrict oneself to subspaces H of $\ell^2(\mathsf{E})$ that give spanning trees or forests. For example, if G is a graph that is embedded on a surface (such as a punctured plane or the 2-torus), let H be the orthocomplement of the boundaries (i.e., of the image of the boundary operator ∂_2). In this case, the measure $\mathbf{P}^{H}$ is the uniform measure on maximal subgraphs that do not contain any boundary in the sense that no linear combination of the edges is a boundary. (This alternative description is proved by the considerations of the following paragraph.) Properties of $\mathbf{P}^{H}$ are worth investigation. A particular example arises as follows. If a graph is embedded on a torus and one takes a uniform spanning tree of the graph, then its complement on the dual graph relative to the surface contains only noncontractible cycles. The distribution on homology is worth investigation. Can we calculate the distribution of the (unsigned) homology basis it gives? Does it have a limit as the mesh of the graph tends to 0? Presumably the limit does exist and is conformally invariant, but it would greatly help if one could find a way to generate the homology basis directly without generating the entire subgraph, such as via some algorithm analogous to those of Aldous/Broder or Wilson.

For another class of examples, consider a finite CW-complex K of dimension d. Given $0 < k \le d$, the representation of the matroid corresponding to the matrix (with respect to the usual cellular bases) of the boundary operator ∂_k from k-chains to $(k-1)$-chains yields a probability measure $\mathbf{P}_k^\partial$ on the set of maximal k-subcomplexes L of K with $\mathrm{H}_k(\mathrm{L}; \mathbf{Q}) = 0$. For example, $\mathbf{P}_1^\partial$ is the uniform spanning tree on the 1-skeleton of K. In general, as shown by Lyons (2003), the probability of such a subcomplex L is proportional to the square of the order of the torsion subgroup of $\mathrm{H}_{k-1}(\mathrm{L}; \mathbf{Z})$, the $(k-1)$-dimensional homology group of the subcomplex L. When K is a simplex, Kalai (1983) showed that the number of maximal $\mathbf{Q}$-acyclic k-subcomplexes of K counted with these weights is

$$n^{\binom{n-2}{k}},$$

thereby generalizing Cayley's theorem. In case K is infinite and locally finite, one can take free and wired limits analogous to the FSF and the WSF. In Lyons (2003), it is shown that in any amenable transitive contractible complex, these free and wired measures agree, which provides a new proof of a theorem of Cheeger and Gromov (1986) concerning the vanishing of ℓ^2-Betti numbers.

Now specialize to the natural d-dimensional CW-complex determined by the hyperplanes of $\mathbf{R}^d$ passing through points of $\mathbf{Z}^d$ and parallel to the coordinate hyperplanes (so the 0-cells are the points of $\mathbf{Z}^d$). In Lyons (2003), it is shown that the $\mathbf{P}_k^\partial$-probability that a given k-cell belongs to the random k-subcomplex is k/d. But many questions are open. Among the most important are the following two:

- What is the $(k-1)$-dimensional (co)homology of the k-subcomplex? In the case $k = 1$ of spanning forests, this asks how many trees there are, the question answered by Pemantle (1991).
- If one takes the 1-point compactification of the subcomplex, what is the k-dimensional (co)homology? In the case of spanning forests, this asks how many ends there are in the tree(s), the question answered partially by Pemantle (1991) and completely by BLPS (2001).

Note that by translation-invariance of (co)homology and ergodicity of $\mathbf{P}_k^\partial$, we have that the values of the (co)homology groups are constants a.s.

The two questions above are interesting even for rational (co)homology. It then follows trivially from the Alexander duality theorem and the results of Pemantle (1991) and BLPS (2001) that for $k = d - 1$, we have $\mathrm{H}_{k-1}(\mathrm{L}; \mathbf{Q}) = 0$ $\mathbf{P}_k^\partial$-a.s., while $\mathbf{P}_k^\partial$-a.s. $\mathrm{H}^k(\mathrm{L} \cup \infty; \mathbf{Q})$ is 0 for $2 \le d \le 4$ and is (naturally isomorphic to) an infinite direct product of $\mathbf{Q}$ for $d \ge 5$ (so the homology is the infinite direct sum of $\mathbf{Q}$). It also follows from the Alexander duality theorem and from equality of free and wired limits that if $d = 2k$, then the a.s. values of $H^k(L \cup \infty; \mathbf{Q})$ and $H_{k-1}(L; \mathbf{Q})$ are the same (naturally isomorphic), so that the two bulleted questions above are dual in that case. Since even the finite complexes can have nontrivial integral homology, it is probably more interesting to examine the quotient of integral cohomology by integral cohomology with compact support. In the present case, the finite complexes have finite groups $H_{k-1}(L; \mathbf{Z})$, so we might simply ask about the Betti numbers in the infinite limit.

Many matroids, of course, are not representable. For them, the above theory gives no measure on the bases. A first clue of how to define an interesting measure nevertheless comes from the following observation. Suppose we choose a uniform spanning tree from a graph that has n vertices. If we then choose an edge uniformly from the tree, the chance of picking e is $Y(e, e)/(n-1)$ by Kirchhoff's Theorem, where Y is the transfer current matrix. Therefore $\sum_{e \in E} Y(e, e) = n - 1$, a theorem of Foster (1948) on electrical networks. One might thus expect something interesting from the measures we now introduce, even when specialized to a graphical matroid.

A second clue is that every matroid $\mathscr{M} = (E, \mathscr{B})$ has naturally associated to it a simplicial complex $K_{\mathscr{M}}$ formed from its independent sets, where a subset of E is called **independent** if it lies in some base.

We now see how to define a natural probability measure on $\mathscr{B}$. Namely, let r be the rank of $\mathscr{M}$, which is one more than the dimension of $K_{\mathscr{M}}$. The boundary operator ∂_{r-1} for $K_{\mathscr{M}}$ gives, as above, a probability measure $\mathbf{P}^{\partial}_{r-1}$ on collections of bases, and then one may choose uniformly an element of such a collection to obtain, finally, a probability measure $\mathbf{P}^{\mathrm{cx}}_{r-1}$ on $\mathscr{B}$. This is not just a complicated way of defining the uniform measure on $\mathscr{B}$, yet it is a measure that is invariant under automorphisms of the matroid. Because of this, it gives a new measure even for the graphic matroid, i.e., a new measure on spanning trees of a graph that, although not uniform, is invariant under automorphisms of the graph, as well as under all matroid automorphisms. Thus, this new measure reflects more of the structure of the graph and of the matroid than does the uniform measure. If one does not use the top dimension $r-1$ but a dimension $k < r - 1$, then one obtains an automorphism-invariant probability measure $\mathbf{P}^{\mathrm{cx}}_k$ on the independent sets of cardinality k.

- How are these measures $\mathbf{P}^{\mathrm{cx}}_k$ related to each other?
- How do they behave under the standard matroid operations?
- Can one describe the measures more explicitly and directly for graphic matroids?

13. Open questions: Dynamical systems

Let Γ be a countable infinite discrete abelian group, such as $\mathbf{Z}^n$. Let $\widehat{\Gamma}$ be the group dual to Γ, a compact group equal to $\mathbf{R}^n/\mathbf{Z}^n$ when $\Gamma = \mathbf{Z}^n$. If $f : \widehat{\Gamma} \to [0, 1]$ is a measurable function, then multiplication by f is a positive contraction on

$L^2_{\mathbf{C}}(\widehat{\Gamma}, \lambda)$, where λ is unit Haar measure. Since $L^2_{\mathbf{C}}(\widehat{\Gamma}, \lambda)$ is isomorphic to $\ell^2(\Gamma; \mathbf{C})$ via the Fourier transform, there is an associated probability measure $\mathbf{P}^f$ on 2^Γ. As an easy example, if f is a constant, p, then $\mathbf{P}^f$ is just the Bernoulli(p) process on Γ. In general, the measure $\mathbf{P}^f$ is invariant under the natural Γ action and has a trivial full tail σ-field by Theorem 7.15. Lyons and Steif (2003) have shown that if $\Gamma = \mathbf{Z}^n$, then for any f, we also have that the dynamical system $(2^\Gamma, \mathbf{P}^f, \Gamma)$ is a Bernoulli shift, i.e., is isomorphic to an i.i.d. process. Therefore, by Ornstein's theorem (and its generalizations, see Katznelson and Weiss (1972), Conze (1972/73), Thouvenot (1972), and Ornstein and Weiss (1987)), it is characterized up to isomorphism by its entropy.

- What is the entropy of the dynamical system $(2^\Gamma, \mathbf{P}^f, \Gamma)$?
- We conjecture that entropy is concave. In other words, if $h(f)$ denotes the entropy of the dynamical system $(2^\Gamma, \mathbf{P}^f, \Gamma)$, then $h\big((f+g)/2\big) \ge \big(h(f)+h(g)\big)/2$ for all f and g. This is a corollary of Conjecture 9.2, and even from a restricted version of (9.1) that assumes that both Q_1 and Q_2 are Toeplitz matrices.
- If $0 \le f \le g \le 1$, then there is a monotone coupling of $\mathbf{P}^f$ and $\mathbf{P}^g$ by Theorem 8.1. Can an explicit monotone coupling be given? For example, if $f = pg$, where p is a constant, then this can be done by using the fact that $\mathbf{P}^{pg}$ has the same law as the pointwise minimum of independent processes $\mathbf{P}^p$ and $\mathbf{P}^g$.
- Consider the case $\Gamma = \mathbf{Z}$. Note that translation and flip of f yields the same measure $\mathbf{P}^f$, even though f changes. Does $\mathbf{P}^f$ determine f up to translation and flip?

Additional questions concerning these systems appear in Lyons and Steif (2003).

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R. L.
Department of Mathematics,
Indiana University,
Bloomington, IN 47405-5701, USA
rdlyons@indiana.edu
http://mypage.iu.edu/~rdlyons/

and

School of Mathematics,
Georgia Institute of Technology,
Atlanta, GA 30332-0160, USA

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